ON CYCLIC STAR-AUTONOMOUS CATEGORIES

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ABSTRACT. We discuss *cyclic star-autonomous categories*; that is, unbraided star-autonomous categories in which the left and right duals of every object p are linked by coherent natural isomorphism. We settle coherence questions which have arisen concerning such cyclicity isomorphisms, and we show that such cyclic structures are the natural setting in which to consider enriched profunctors. Specifically, if $\mathcal{V}$ is a cyclic star-autonomous category, then the collection of $\mathcal{V}$-enriched profunctors carries a canonical cyclic structure. In the case of braided star-autonomous categories, we discuss the correspondences between cyclic structures and balances or tortile structures. Finally, we show that every cyclic star-autonomous category is equivalent to one in which the cyclicity isomorphisms are identities.

1. Introduction (Overview)

In an arbitrary star-autonomous category (in particular, one which is not necessarily symmetric [Bar95]), every object p has two duals: commonly, one is denoted p^* or $p^\perp$, the other ${}^*p$ or ${}^\perp p$. (We prefer * over $^\perp$ except, for obscure reasons, in the posetal examples below.) A *cyclic star-autonomous poset* is defined in [Yet90] to be a star-autonomous poset with the property that these two duals always coincide. It is important to note that, even in the posetal case, cyclicity is a much weaker phenomenon than symmetry.

1.1. EXAMPLE. It is well-known that any ordered group $\mathfrak{g} = ((g, \leq), \cdot, \eta, (\ )^{-1})$ determines a closed monoidal poset with $\alpha \multimap \beta := \alpha^{-1} \cdot \beta$ and $\beta \circ\!\!- \alpha := \beta \cdot \alpha^{-1}$; it follows that every element of the group is *dualising*. Thus every *pointed ordered group* $(\mathfrak{g}, \varepsilon)$, for ε a fixed but arbitrary element of g, determines a star-autonomous poset. (This example is called an *ordered shift group* in [CS97].) Since $\omega^\perp = \omega \multimap \varepsilon$ and ${}^\perp\omega = \varepsilon \circ\!\!- \omega$, we have that $(\mathfrak{g}, \varepsilon)$ is cyclic if and only if ε is central. In particular, $(\mathfrak{g}, \eta)$ is always cyclic—but it is symmetric if and only if $\mathfrak{g}$ is abelian.

1.2. EXAMPLE. Binary relations $s \nrightarrow s$ (where s is some fixed but arbitrary set) form a cyclic star-autonomous poset: the tensor product is the usual composition of relations, and the dualising object is the complement of the equality relation $(\neq)$. It is routine to verify that

$$\omega^\perp = (\omega \multimap \neq) = \neg\omega^{\mathsf{rev}} = (\neq \circ\!\!- \omega) = {}^\perp\omega$$

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holds for every $\omega : s \nrightarrow s$. But, of course, symmetry can only occur when $\#s \leq 1$. (Here, ω^{rev} denotes the *reverse* of ω which is more commonly denoted ω^{op}.)

More generally, 2-valued profunctors $\mathbb{p} \nrightarrow \mathbb{p}$ (where $\mathbb{p} = (p, \leq)$, some fixed but arbitrary poset) form a cyclic star-autonomous poset: the tensor product is the usual composition of profunctors, and the dualising object is the complement of the reverse ordering ($\not\geq$). It is again routine to verify that

$$\omega^{\perp} = (\omega \multimap \not\geq) = \neg\omega^{\mathsf{rev}} = (\not\geq \multimapinv \omega) = {}^{\perp}\omega$$

holds for every $\omega : \mathbb{p} \nrightarrow \mathbb{p}$. Again, symmetry can only occur when $\#\mathbb{p} \leq 1$. (Observe that, in general, neither $\neg\omega$ nor ω^{rev} is a profunctor $\mathbb{p} \nrightarrow \mathbb{p}$, but that $\neg\omega^{\mathsf{rev}}$ is.)

It is well-understood, at least in principle, that the term *cyclic star-autonomous category* should mean a star-autonomous category equipped with a coherent natural isomorphism $p^* \longrightarrow {}^*p$. But this raises the question: what are the right coherence axioms? This question is complicated by the fact that there is a second approach to the phenomenon of cyclicity which does not explicitly refer to dual objects. (This is not a new observation; on the contrary, the origin of the term *cyclic* is tied up in this approach—see again, [Yet90].)

Since there are several equivalent definitions of star-autonomous category, let use make clear that we use the one advocated in [CS97]: a linearly distributive category with chosen left and right duals for every object. We generally use $\owedge$ for *tensor* and $\ovee$ for *par*; the linear distributions $q \owedge (s \ovee t) \longrightarrow (q \owedge s) \ovee t$ and $(p \ovee q) \owedge s \longrightarrow p \ovee (q \owedge s)$ are denoted $\vec{\kappa}$ and $\overleftarrow{\kappa}$, respectively.

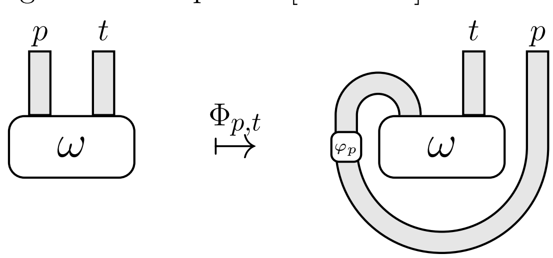
1.3. Remark. Let $\mathcal{K} = (\mathcal{K}, \owedge, e, \ovee, d, (\ )^*, {}^*(\ ))$ be a star-autonomous category and let $\langle s, t\rangle_{\mathcal{K}}$ denote the external set of arrows $s \longrightarrow t$; then natural isomorphisms of the form $p^* \longrightarrow {}^*p$ are in bijective correspondence with those of the form $\langle p \owedge t, d\rangle_{\mathcal{K}} \longrightarrow \langle t \owedge p, d\rangle_{\mathcal{K}}$.

We shall denote this correspondence (summarised below) by a change of case: lower case for natural isomorphisms of the first form, upper case for those of the second.

$$\begin{tikzcd} & t \arrow[dl, "\mathsf{lCurry}(\omega)"'] \arrow[dr, "\mathsf{rCurry}(\psi)"] & \\ p^* \arrow[rr, "\varphi_p"'] & & {}^*p \end{tikzcd} \qquad \Leftrightarrow \qquad \begin{tikzcd} p \owedge t \arrow[d, "\omega"] & & t \owedge p \arrow[d, "\psi"] \\ d \arrow[r, phantom, "\overset{\Phi_{p,t}}{\mapsto}"] & & d \end{tikzcd}$$

That is, given φ, we set $\Phi_{p,t}(\omega) = \mathsf{rCurry}^{-1}(\mathsf{lCurry}(\omega)\ ;\ \varphi_p)$; and, given Φ, we set $\varphi_p = \mathsf{rCurry}(\Phi_{p,p^*}(\mathsf{lCurry}^{-1}(\mathrm{id}_{p^*})))$.

Half of this correspondence can be easily depicted in terms of the graphical calculus for star-autonomous categories developed in [BCST96].



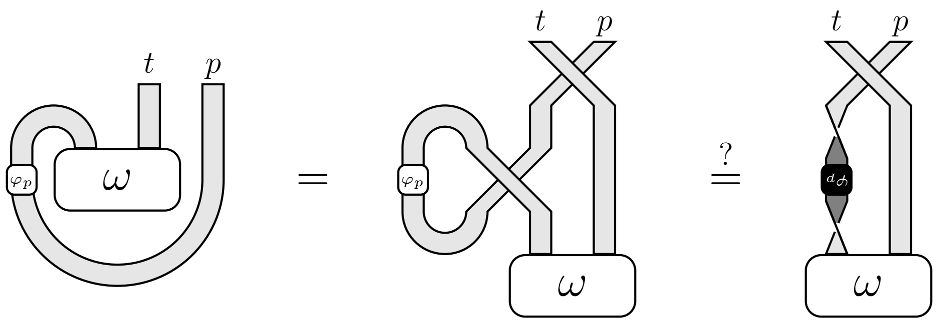
(Our reason for using *ribbons/tape* rather than *strings/wires* shall soon become apparent.)

Dually, there is also a bijective correspondence with natural isomorphisms of the form $\langle e, t \varovee p \rangle_{\mathcal{K}} \longrightarrow \langle e, p \varovee t \rangle_{\mathcal{K}}$; but we shall have no occasion to use this in the present paper. (It will, however, play a prominent role in [EM10].)

So, a cyclic star-autonomous category could be defined either as a pair $(\mathcal{K}, \varphi)$ with some coherence axioms for φ, or as a pair $(\mathcal{K}, \Phi)$ with some coherence axioms for Φ. Rosenthal [Ros94] takes the former approach; Blute, Lamarche and Ruet [BLR02] the latter. In section 2, we show that these two definitions are inequivalent; we further introduce notions of $\varowedge$*-semicyclicity* and $\varovee$*-semicyclicity*, each of which lies between the stronger notion of cyclicity (that of [BLR02]) and the weaker one (that of [Ros94]). Henceforth, we refer to the weaker notion as *quasicyclicity*, so as to reserve the term *cyclicity* for the stronger notion (which is also the conjunction of $\varowedge$-semicyclicity and $\varovee$-semicyclicity).

Recall that, if $\mathcal{V}$ is a complete and cocomplete, symmetric and closed monoidal category, and $\mathbb{c}$ is a small $\mathcal{V}$-category, then the monoidal category $\mathcal{C} = \mathfrak{Prof}_{\mathcal{V}}(\mathbb{c}, \mathbb{c})$ is also closed, but not (in general) symmetric[1]. Rosenthal, doubtless inspired by Example 1.2, observed that: if $\mathcal{V}$ also admits a dualising object, then so does $\mathcal{C}$; and, moreover, that the resultant star-autonomous structure on $\mathcal{C}$ is always quasicyclic [Ros94]. We generalise these results in section 3: to derive a star-autonomous structure on $\mathcal{C}$ it suffices that $\mathcal{V}$ be $\varowedge$-semicyclic (and $\mathcal{C}$ is also $\varowedge$-semicyclic in this case); moreover, if $\mathcal{V}$ is also quasicyclic (and therefore cyclic), then the same is true of $\mathcal{C}$.

The reader would be quickly forgiven for assuming that a braided star-autonomous category is necessarily cyclic—after all, one would normally set up the structure of such a category in such a way as to have $p^* = {}^*p$, and it is difficult at first to imagine that an identity natural transformation could fail to be coherent. But the essential import of braidedness is that our graphical calculus should no longer be restricted to the plane; therefore, we should be able to pull ω over the p-ribbon in (our depiction of) $\Phi_{p,t}(\omega)$ as follows.



The second (questionable) step is pulling the p-ribbon straight. (We naturally use *reverse video* to depict the opposite side of a ribbon or box.) Even if φ were chosen to be an identity, this step results in a 2π-twist on the ribbon p—so it should not be too surprising to learn that one cannot have a cyclicity on a braided star-autonomous category unless it

[1] Closedness follows from [Day70], since it is possible to construct a *promonoidal* $\mathcal{V}$*-category* $\mathbb{c}^{\mathsf{op}} \otimes \mathbb{c}$ such that $\mathcal{V}^{\mathbb{c}^{\mathsf{op}} \otimes \mathbb{c}}$ and $\mathfrak{Prof}_{\mathcal{V}}(\mathbb{c}, \mathbb{c})$ are equivalent as monoidal categories.

also carries a *balance* [JS91]. In fact, this relationship between cyclicity and balancedness is quite well-known among people who study *compact* star-autonomous categories—see, for example, [Yet92, Mal95]. We show that this relationship does not depend on compactness in section 4; we further show that even quasicyclicity is not guaranteed for a braided star-autonomous category by developing a corresponding notion of *quasibalance*.

If the reader wonders why φ_p appears so puny in the figures above, it is because of the strictification result proven in section 5. Let us say that an arbitrary star-autonomous category has *strict negations* if all of the de Morgan isomorphisms

$$
\begin{array}{cccc}
p^* \oslash q^* \longrightarrow (q \oslash p)^* & (p \oslash q)^* \longrightarrow q^* \oslash p^* & e \longrightarrow d^* & e^* \longrightarrow d \\
{}^*p \oslash {}^*q \longrightarrow {}^*(q \oslash p) & {}^*(p \oslash q) \longrightarrow {}^*q \oslash {}^*p & e \longrightarrow {}^*d & {}^*e \longrightarrow d
\end{array}
$$

(which are all denoted ϑ) and the cancellation isomorphisms

$$
p \longrightarrow ({}^*p)^* \qquad\qquad p \longrightarrow {}^*(p^*)
$$

(both denoted η) are identities, and that a cyclic star-autonomous category has *a strict negation* if, in addition, φ is the identity natural transformation. Then every star-autonomous category is equivalent (as a linearly distributive category, and therefore also as a star-autonomous category) to one with strict negations, and every cyclic star-autonomous category is equivalent to one with a strict negation.

We foreshadow the former result (which, obviously, does not depend on any of the results of sections 2–4) by suppressing the relevant isomorphisms in the graphical calculus for arbitrary star-autonomous categories; in effect, we allow strings to be relabelled "on the fly". The latter result entails that the graphical calculus for cyclic star-autonomous categories should also suppress components of φ; in effect, it should be identical to that for arbitrary star-autonomous categories, except that a larger number of string relabellings are permitted.

2. Coherence axioms

Throughout this section: $\mathcal{K} = (\mathcal{K}, \oslash, e, \owedge, d, (\ )^*, {}^*(\ ))$ denotes a star-autonomous category; φ, a natural isomorphism $p^* \longrightarrow {}^*p$; and Φ the corresponding natural isomorphism $\langle p \oslash t, d\rangle_{\mathcal{K}} \longrightarrow \langle t \oslash p, d\rangle_{\mathcal{K}}$. Figure 1 lists a number of possible coherence axioms for φ; Figure 2 does the same for Φ. (In the latter figure, lbind and rbind refer to the functions

$$
\langle (p \owedge q) \oslash (s \oslash t), d\rangle_{\mathcal{K}} \longleftarrow \langle p \oslash t, d\rangle_{\mathcal{K}} \times \langle q \oslash s, d\rangle_{\mathcal{K}} \longrightarrow \langle (p \oslash q) \oslash (s \owedge t), d\rangle_{\mathcal{K}}
$$

which map a pair of arrows (ω, ψ) to the composites

$$
\begin{array}{c}
(p \ovee q) \owedge (s \owedge t) \\
{\scriptstyle \alpha^{-1}}\big\downarrow \\
((p \ovee q) \owedge s) \owedge t \\
{\scriptstyle \overleftarrow{\kappa} \owedge \mathrm{id}}\big\downarrow \\
(p \ovee (q \owedge s)) \owedge t \\
{\scriptstyle (\mathrm{id} \ovee \psi) \owedge \mathrm{id}}\big\downarrow \\
(p \ovee d) \owedge t \\
{\scriptstyle \rho \owedge \mathrm{id}}\big\downarrow \\
p \owedge t \\
{\scriptstyle \omega}\big\downarrow \\
d
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
(p \owedge q) \owedge (s \ovee t) \\
\big\downarrow{\scriptstyle \alpha} \\
p \owedge (q \owedge (s \ovee t)) \\
\big\downarrow{\scriptstyle \mathrm{id} \owedge \vec{\kappa}} \\
p \owedge ((q \owedge s) \ovee t) \\
\big\downarrow{\scriptstyle \mathrm{id} \owedge (\psi \ovee \mathrm{id})} \\
p \owedge (d \ovee t) \\
\big\downarrow{\scriptstyle \mathrm{id} \owedge \lambda} \\
p \owedge t \\
\big\downarrow{\scriptstyle \omega} \\
d
\end{array}
$$

respectively.)

In [Ros94], a cyclic star-autonomous category is defined to be a pair $(\mathcal{K}, \varphi)$ such that φ satisfies (R). Rosenthal proves that this axiom suffices to prove that, for every pair of even integers m and n (and every pair of odd integers m and n), there is a unique isomorphism $p^{*m} \longrightarrow p^{*n}$ built up from the components of φ.

However, in [BLR02], a cyclic star-autonomous category is defined to be a pair $(\mathcal{K}, \Phi)$ such that Φ satisfies (BLR_0) and (BLR_2). Blute, Lamarche and Ruet prove: that (BLR_0) is equivalent to (T_0); that (R) is equivalent to (INV); and, that the latter follows from (BLR_2). They further conjecture that their definition is strictly stronger than that of Rosenthal.

2.1. Example. Consider $(\mathbf{Vec}_{\mathrm{fd}}, \otimes_k, k)$, where k is a field not of characteristic two; since $\otimes_k$ is symmetric, we can (and do) choose to define $(\ )^*$ and ${}^*(\ )$ so that $p^* = {}^*p$ for all spaces p. So each non-zero scalar determines a natural isomorphism $p^* \longrightarrow {}^*p$; the latter satisfies (R) if and only if the scalar is ± 1, and (T_0) if and only if the scalar is 1.

We shall find it convenient to consider yet more possible (combinations of) axioms, as follows.

2.2. Definitions. *We call* φ*: a* $\ovee$-semicycle*, if it satisfies* (P_2)*;* quasicycle*, if it satisfies* (R)*;* $\owedge$-semicycle*, if it satisfies* (T_2)*;* cycle*, if it is both a* $\owedge$*-semicycle and a* $\ovee$*-semicycle.*

The pair $(\mathcal{K}, \varphi)$ *is called a* ($\ovee$-semi-, quasi-, $\owedge$-semi-)cyclic star-autonomous category *whenever* φ *is a (*$\ovee$*-semi-, quasi-,* $\owedge$*-semi-)cycle.*

2.3. Lemma. *The following dependencies exist between the axioms listed in Figure 1.*

$$
\begin{array}{rcl}
(\mathrm{T}_2) & \Rightarrow & (\mathrm{T}_0) \\
(\mathrm{P}_2) & \Rightarrow & (\mathrm{P}_0) \\
(\mathrm{R}) & \Rightarrow & ((\mathrm{T}_0) \Leftrightarrow (\mathrm{P}_0))
\end{array}
\qquad\qquad
\begin{array}{rcl}
(\mathrm{T}_2) & \Rightarrow & ((\mathrm{P}_0) \Leftrightarrow (\mathrm{R})) \\
(\mathrm{P}_2) & \Rightarrow & ((\mathrm{T}_0) \Leftrightarrow (\mathrm{R})) \\
(\mathrm{R}) & \Rightarrow & ((\mathrm{T}_2) \Leftrightarrow (\mathrm{P}_2))
\end{array}
$$

In particular, a cycle is a quasicycle; moreover, each of the following four pairs of axioms is equivalent to cyclicity.

$$
\{(\mathrm{P}_0), (\mathrm{T}_2)\} \qquad \{(\mathrm{R}), (\mathrm{T}_2)\} \qquad \{(\mathrm{P}_2), (\mathrm{R})\} \qquad \{(\mathrm{P}_2), (\mathrm{T}_0)\}
$$

$$
\begin{tikzcd}[column sep=small]
d^* \arrow[rr, "\varphi_d"] \arrow[dr, "\vartheta"'] \arrow[rr, phantom, shift right=4ex, "(\mathrm{P}_0)"] & & {}^*d \arrow[dl, "\vartheta"] \\
& e &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& r \arrow[dl, "\eta"'] \arrow[dr, "\eta"] \arrow[d, phantom, "(\mathrm{R})"] & \\
({}^*r)^* \arrow[r, "(\varphi_r)^*"'] & (r^*)^* \arrow[r, "\varphi_{r^*}"'] & {}^*(r^*)
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
e^* \arrow[rr, "\varphi_e"] \arrow[dr, "\vartheta"'] \arrow[rr, phantom, shift right=4ex, "(\mathrm{T}_0)"] & & {}^*e \arrow[dl, "\vartheta"] \\
& d &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(p \ovee q)^* \arrow[r, "\varphi_{p \ovee q}"] \arrow[d, "\vartheta"'] \arrow[dr, phantom, "(\mathrm{P}_2)"] & {}^*(p \ovee q) \arrow[d, "\vartheta"] \\
q^* \owedge p^* \arrow[r, "\varphi_q \owedge \varphi_p"'] & {}^*q \owedge {}^*p
\end{tikzcd}
\qquad
\begin{tikzcd}
(p \owedge q)^* \arrow[r, "\varphi_{p \owedge q}"] \arrow[d, "\vartheta"'] \arrow[dr, phantom, "(\mathrm{T}_2)"] & {}^*(p \owedge q) \arrow[d, "\vartheta"] \\
q^* \ovee p^* \arrow[r, "\varphi_q \ovee \varphi_p"'] & {}^*q \ovee {}^*p
\end{tikzcd}
$$

Figure 1: Axioms for a φ

Proof. We discuss only the first row of assertions: the second row of assertions are dual; and, although the third row of assertions (which are, in any case, much less surprising) can be proven directly, it is simpler to see them as a corollary of Lemma 2.4 and its dual.

That $(\mathrm{T}_2) \Rightarrow (\mathrm{T}_0)$ follows from a more general fact—namely, that a "semigroupal" natural isomorphism between strong monoidal functors is necessarily monoidal (see Lemma A.1). It should be clear that (T_2) asserts that φ is semigroupal, and that (T_2) together with (T_0) asserts that φ is monoidal, with respect to strong monoidal functors $(\mathcal{K}, \owedge, e) \longrightarrow (\mathcal{K}, \ovee, d)$ overlying $(\ )^*$ and ${}^*(\ )$.)

To prove $(\mathrm{T}_2) \Rightarrow ((\mathrm{P}_0) \Leftrightarrow (\mathrm{R}))$, it will be convenient to derive a form of (T_2) which does not explicitly refer to $\ovee$. This is achieved by applying the natural isomorphisms $(x \multimap z) \xrightarrow{\sim} x^* \ovee z$ and $(z \multimapinv y) \xrightarrow{\sim} z \ovee {}^*y$, as follows.

$$
\begin{tikzcd}[column sep=small]
(p \owedge q)^* \arrow[rr, "\vartheta"'] \arrow[rrrrrrrr, bend left, "\varphi_{p \owedge q}"] \arrow[dr, "\sim"] & & q^* \ovee p^* \arrow[rr, "\mathrm{id}_q \ovee \varphi_p"'] \arrow[dl, "\sim"'] & & q^* \ovee {}^*p \arrow[rr, "\varphi_q \ovee \mathrm{id}_p"'] \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & & {}^*q \ovee {}^*p \arrow[rr, "\vartheta"'] \arrow[dr, "\sim"] & & {}^*(p \owedge q) \\
& q \multimap (p^*) \arrow[r, "\mathrm{id}_q \multimap \varphi_p"'] & q \multimap ({}^*p) \arrow[rr, "\sim"'] & & (q^*) \multimapinv p \arrow[r, "\varphi_q \multimapinv \mathrm{id}_p"'] & ({}^*q) \multimapinv p \arrow[urr, "\sim"'] & & &
\end{tikzcd}
$$

The outer hexagon of the diagram above forms the central cell in the diagram below, (But we have replaced $\psi \multimap \omega$ and $\omega \multimapinv \psi$ by their more colloquial forms: $\psi \circ (\ ) \circ \omega$ and $\omega\ ;\ (\ )\ ;\ \psi$, respectively.) The two cells labelled (N) are naturality squares, and all the unlabelled cells (including the outermost square) are tautologies that hold in arbitrary star-autonomous categories.

$$
\begin{tikzcd}
e \arrow[rr, equals] \arrow[ddd, equals] & & e \arrow[rr, equals] \arrow[dl, "\vartheta"'] \arrow[dr, "\vartheta"] \arrow[d, phantom, "(?)"] & & e \arrow[ddd, "\ulcorner \mathrm{id}_p \urcorner"] \\
& d^* \arrow[rr, "\varphi_d"] \arrow[d, "\gamma^*"'] \arrow[drr, phantom, "(\mathrm{N})"] & {} & {}^*d \arrow[d, "{}^*\gamma"] & \\
& (p \parr p^*)^* \arrow[rr, "\varphi_{p \parr p^*}"] \arrow[d, "\sim"'] & & {}^*(p \parr p^*) & \\
e \arrow[r, "\ulcorner \mathrm{id}_{p^*} \urcorner"] \arrow[d, "\ulcorner \mathrm{id}_{{}^*p} \urcorner"'] \arrow[dr, "\ulcorner \varphi_p \urcorner"] & p^* \multimap p^* \arrow[d, "(\ ) \,;\, \varphi_p"] \arrow[rr, phantom, "(!)"] & & {}^*(p^*) \multimapinv p \arrow[u, "\sim"'] \arrow[dr, phantom, "(??)"] & p \multimapinv p \arrow[l, "\eta \circ (\ )"'] \arrow[d, "\eta \circ (\ )"] \\
{}^*p \multimap {}^*p \arrow[r, "\varphi_p \,;\, (\ )"'] \arrow[rrrr, bend right, "\sim"'] & p^* \multimap {}^*p \arrow[rr, "\sim", "(\mathrm{N})"'] & & (p^*)^* \multimapinv p \arrow[u, "\varphi_{p^*} \circ (\ )"] & ({}^*p)^* \multimapinv p \arrow[l, "(\varphi_p)^* \circ (\ )"]
\end{tikzcd}
$$

The cell labelled (?) equals (P_0). If it holds, then we conclude that

$$\ulcorner \eta \urcorner = \eta \circ (\ulcorner \mathrm{id}_p \urcorner) = \varphi_{p^*} \circ ((\varphi_p)^* \circ (\eta \circ (\ulcorner \mathrm{id}_p \urcorner))) = \ulcorner \varphi_{p^*} \circ (\varphi_p)^* \circ \eta \urcorner$$

—and this is tautologously equivalent to (R).

Conversely, if (R) holds, then so does (??); if we choose p to be either d or e, then all the arrows (including the counit γ) are invertible, and this allows us to conclude that (?) (which is (P_0)) holds. ∎

The main result of this section is that our definition of cyclic star-autonomous category agrees with that of [BLR02]; this is an immediate corollary of the following lemma.

2.4. Lemma. *The following axioms are equivalent:* (T_2), (E), (M_2^{-1}). *Moreover,* (BLR_2) *is equivalent to* (INV) $\wedge$ (E).

Proof. Using the $\otimes$-free version of (T_2) derived in the proof of Lemma 2.3 above, we chase the two diagrams simultaneously: given $\omega \in \langle (p \parr q) \parr t, d \rangle_K$,

$$
\begin{tikzcd}
& & & t \arrow[dlll, bend right, "\mathsf{lCurry}(\omega)"'] \arrow[dll, dotted] \arrow[dl, dotted] \arrow[d, dotted] \arrow[dr, dotted] \arrow[drr, dotted] & & \\
(p \parr q)^* \arrow[r, "\sim"'] & q \multimap (p^*) \arrow[r, "\mathrm{id}_q \multimap \varphi_p"'] & q \multimap ({}^*p) \arrow[r, "\sim"'] & (q^*) \multimapinv p \arrow[r, "\varphi_q \multimapinv \mathrm{id}_p"'] & ({}^*q) \multimapinv p \arrow[r, "\sim"'] & {}^*(p \parr q)
\end{tikzcd}
$$

$$
\begin{tikzcd}
(p \parr q) \parr t \arrow[d, "\omega"'] \arrow[dr, phantom, "\overset{\alpha^{-1}_{p,q,t};(\ )}{\mapsto}"] & p \parr (q \parr t) \arrow[d] \arrow[dr, phantom, "\overset{\Phi_{p,q \parr t}}{\mapsto}"] & (q \parr t) \parr p \arrow[d] \arrow[dr, phantom, "\overset{\alpha^{-1}_{q,t,p};(\ )}{\mapsto}"] & q \parr (t \parr p) \arrow[d] \arrow[dr, phantom, "\overset{\Phi_{q,t \parr p}}{\mapsto}"] & (t \parr p) \parr q \arrow[d] \arrow[dr, phantom, "\overset{\alpha^{-1}_{t,p,q};(\ )}{\mapsto}"] & t \parr (p \parr q) \arrow[d] \\
d & d & d & d & d & d
\end{tikzcd}
$$

$$
\begin{tikzcd}
\langle (p \owedge q) \owedge t, d\rangle_{\mathcal{K}} \arrow[r, "\Phi_{p \owedge q, t}"] \arrow[d, "\alpha^{-1}_{p,q,t}\ ;\ (\ )"'] & \langle t \owedge (p \owedge q), d\rangle_{\mathcal{K}} \arrow[d, "\alpha_{t,p,q}\ ;\ (\ )"] \\
\langle p \owedge (q \owedge t), d\rangle_{\mathcal{K}} \arrow[d, "\Phi_{p, q \owedge t}"'] \arrow[r, phantom, "(\mathrm{BLR}_2)"] & \langle (t \owedge p) \owedge q, d\rangle_{\mathcal{K}} \arrow[d, "\Phi_{t \owedge p, q}"] \\
\langle (q \owedge t) \owedge p, d\rangle_{\mathcal{K}} \arrow[r, "\alpha^{-1}_{q,t,p}\ ;\ (\ )"'] & \langle q \owedge (t \owedge p), d\rangle_{\mathcal{K}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle t \owedge p, d\rangle_{\mathcal{K}} \arrow[d, "(\mathrm{INV})"', "\Phi_{t,p} = \Phi^{-1}_{p,t}"] \\
\langle p \owedge t, d\rangle_{\mathcal{K}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\langle (p \owedge q) \owedge t, d\rangle_{\mathcal{K}} \arrow[r, "\Phi_{p \owedge q, t}"] \arrow[d, "\alpha^{-1}_{p,q,t}\ ;\ (\ )"'] & \langle t \owedge (p \owedge q), d\rangle_{\mathcal{K}} \\
\langle p \owedge (q \owedge t), d\rangle_{\mathcal{K}} \arrow[d, "\Phi_{p, q \owedge t}"'] \arrow[r, phantom, "(\mathrm{E})"] & \langle (t \owedge p) \owedge q, d\rangle_{\mathcal{K}} \arrow[u, "\alpha^{-1}_{t,p,q}\ ;\ (\ )"'] \\
\langle (q \owedge t) \owedge p, d\rangle_{\mathcal{K}} \arrow[r, "\alpha^{-1}_{q,t,p}\ ;\ (\ )"'] & \langle q \owedge (t \owedge p), d\rangle_{\mathcal{K}} \arrow[u, "\Phi_{q, t \owedge p}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle e \owedge t, d\rangle_{\mathcal{K}} \arrow[dr, "\lambda^{-1}_t\ ;\ (\ )"] \arrow[dd, "\Phi_{e,t}"'] & \\
(\mathrm{BLR}_0) & \langle t, d\rangle_{\mathcal{K}} \arrow[dl, "\rho_t\ ;\ (\ )"] \\
\langle t \owedge e, d\rangle_{\mathcal{K}} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\langle p \owedge (s \owedge t), d\rangle_{\mathcal{K}} \arrow[r, "\Phi_{p, s \owedge t}"] \arrow[d, "\alpha_{p,s,t}\ ;\ (\ )"'] & \langle (s \owedge t) \owedge p, d\rangle_{\mathcal{K}} \\
\langle (p \owedge s) \owedge t, d\rangle_{\mathcal{K}} \arrow[d, "\Phi_{p \owedge s, t}"'] \arrow[r, phantom, "(\mathrm{E}^{-1})"] & \langle s \owedge (t \owedge p), d\rangle_{\mathcal{K}} \arrow[u, "\alpha_{s,t,p}\ ;\ (\ )"'] \\
\langle t \owedge (p \owedge s), d\rangle_{\mathcal{K}} \arrow[r, "\alpha_{t,p,s}\ ;\ (\ )"'] & \langle (t \owedge p) \owedge s, d\rangle_{\mathcal{K}} \arrow[u, "\Phi_{t \owedge p, s}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle t \owedge e, d\rangle_{\mathcal{K}} \arrow[dr, "\rho^{-1}_t\ ;\ (\ )"] \arrow[dd, "\Phi_{t,e}"'] & \\
(\mathrm{M}_0) & \langle t, d\rangle_{\mathcal{K}} \arrow[dl, "\lambda_t\ ;\ (\ )"] \\
\langle e \owedge t, d\rangle_{\mathcal{K}} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\langle p \owedge t, d\rangle_{\mathcal{K}} \times \langle q \owedge s, d\rangle_{\mathcal{K}} \arrow[r, "\Phi_{p,t} \times \Phi_{q,s}"] \arrow[dd, "\mathsf{lbind}"'] & \langle t \owedge p, d\rangle_{\mathcal{K}} \times \langle s \owedge q, d\rangle_{\mathcal{K}} \arrow[d, "\sim"] \\
(\mathrm{M}_2) & \langle s \owedge q, d\rangle_{\mathcal{K}} \times \langle t \owedge p, d\rangle_{\mathcal{K}} \arrow[d, "\mathsf{rbind}"] \\
\langle (p \ovee q) \owedge (s \owedge t), d\rangle_{\mathcal{K}} \arrow[r, "\Phi_{p \ovee q, s \owedge t}"'] & \langle (s \owedge t) \owedge (p \ovee q), d\rangle_{\mathcal{K}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\langle p \owedge t, d\rangle_{\mathcal{K}} \times \langle q \owedge s, d\rangle_{\mathcal{K}} \arrow[r, "\Phi_{p,t} \times \Phi_{q,s}"] \arrow[dd, "\mathsf{rbind}"'] & \langle t \owedge p, d\rangle_{\mathcal{K}} \times \langle s \owedge q, d\rangle_{\mathcal{K}} \arrow[d, "\sim"] \\
(\mathrm{M}_2^{-1}) & \langle s \owedge q, d\rangle_{\mathcal{K}} \times \langle t \owedge p, d\rangle_{\mathcal{K}} \arrow[d, "\mathsf{lbind}"] \\
\langle (p \owedge q) \owedge (s \ovee t), d\rangle_{\mathcal{K}} \arrow[r, "\Phi_{p \owedge q, s \ovee t}"'] & \langle (s \ovee t) \owedge (p \owedge q), d\rangle_{\mathcal{K}}
\end{tikzcd}
$$

Figure 2: Axioms for a Φ

should equal

$$
\begin{tikzcd}
 & t \arrow[dl, "\mathsf{lCurry}(\omega)"'] \arrow[dr, dotted] & \\
(p \owedge q)^* \arrow[rr, "\varphi_{p \owedge q}"'] & & {}^*(p \owedge q)
\end{tikzcd}
$$

$$
\begin{tikzcd}
(p \owedge q) \owedge t \arrow[d, "\omega"'] & & t \owedge (p \owedge q) \arrow[d] \\
d & \overset{\Phi_{p \owedge q, t}}{\longmapsto} & d
\end{tikzcd}
$$

respectively.

Now, if (E) holds, then (M_2^{-1}) can be verified by chasing a pair (ω, ψ) through the diagram, as follows.

$$
\begin{aligned}
&\Phi_{p \owedge q, s \ovee t}(\mathsf{rbind}(\omega, \psi)) \\
&\quad = \alpha^{-1}_{s \ovee t, p, q} \; ; \; \Phi_{q, (s \ovee t) \owedge p}(\alpha^{-1}_{q, s \ovee t, p} \; ; \; \Phi_{p, q \owedge (s \ovee t)}(\alpha^{-1}_{p, q, s \ovee t} \; ; \; \mathsf{rbind}(\omega, \psi))) && (1) \\
&\quad = \alpha^{-1}_{s \ovee t, p, q} \; ; \; \Phi_{q, (s \ovee t) \owedge p}(\alpha^{-1}_{q, s \ovee t, p} \; ; \; \Phi_{p, q \owedge (s \ovee t)}(\mathrm{id}_p \owedge (\vec{\kappa}_{q,s,t} \; ; \; \psi \ovee \mathrm{id}_t \; ; \; \lambda_t) \; ; \; \omega)) && (2) \\
&\quad = \alpha^{-1}_{s \ovee t, p, q} \; ; \; \Phi_{q, (s \ovee t) \owedge p}(\alpha^{-1}_{q, s \ovee t, p} \; ; \; (\vec{\kappa}_{q,s,t} \; ; \; \psi \ovee \mathrm{id}_t \; ; \; \lambda_t) \owedge \mathrm{id}_p \; ; \; \Phi_{p,t}(\omega)) && (3) \\
&\quad = \alpha^{-1}_{s \ovee t, p, q} \; ; \; \Phi_{q, (s \ovee t) \owedge p}(\mathrm{id}_q \owedge (\overleftarrow{\kappa}_{s,t,p} \; ; \; \mathrm{id}_s \ovee \Phi_{p,t}(\omega) \; ; \; \rho_s) \; ; \; \psi) && (4) \\
&\quad = \alpha^{-1}_{s \ovee t, p, q} \; ; \; (\overleftarrow{\kappa}_{s,t,p} \; ; \; \mathrm{id}_s \ovee \Phi_{p,t}(\omega) \; ; \; \rho_s) \owedge \mathrm{id}_q \; ; \; \Phi_{q,s}(\psi) && (5) \\
&\quad = \mathsf{lbind}(\Phi_{q,s}(\psi), \Phi_{p,t}(\omega)) && (6)
\end{aligned}
$$

(Equation (1) applies (E); Equations (2) and (6), the definitions of rbind and lbind, respectively; Equations (3) and (5), the naturality of Φ. Equation (4) is a simple exercise in linearly distributive category theory—see Lemma A.2.)

Conversely, we can derive (T_2) from (M_2^{-1}), by applying the latter to the pair $\omega = \gamma_p = \mathsf{lCurry}^{-1}(\mathrm{id}_{p^*})$ and $\psi = \gamma_q = \mathsf{lCurry}^{-1}(\mathrm{id}_{q^*})$. (Note that $\mathsf{rbind}(\gamma_p, \gamma_q) \cong \gamma_{p \owedge q}$, as arrows, via the de Morgan isomorphism $\vartheta : (p \owedge q)^* \longrightarrow q^* \ovee p^*$.)

Finally, it is clear that ((INV) $\wedge$ (E)) $\Rightarrow$ (BLR_2) and ((BLR_2) $\wedge$ (INV)) $\Rightarrow$ (E). But since (BLR_2) $\Rightarrow$ (INV) the latter can be sharpened to (BLR_2) $\Rightarrow$ ((E) $\wedge$ (INV)). ∎

For the sake of completeness, we note without proof that the axioms (P_0) and (M_0) are equivalent, as are (P_2), (M_2) and (E^{-1}). Note that (M_0) is (BLR_0)-for-Φ^{-1}, (E^{-1}) is (E)-for-Φ^{-1}, and (M_2^{-1}) is (M_2)-for-Φ^{-1}. Hence, (INV) $\Rightarrow$ ((M_0) $\Leftrightarrow$ (BLR_0)) and (INV) $\Rightarrow$ ((M_2) $\Leftrightarrow$ (M_2^{-1})) are trivial.

3. Enriched profunctors and cyclicity

Throughout this section: $\mathcal{V} = (\mathcal{V}, \owedge, e, \ovee, d, (\;)^*, {}^*(\;), \varphi)$ denotes a $\owedge$-semicyclic star-autonomous category; and, when we speak of $\mathcal{V}$-categories ($\mathcal{V}$-profunctors, *etc.*), then we mean $(\mathcal{V}, \owedge, e)$-categories (resp., $(\mathcal{V}, \owedge, e)$-profunctors, *etc.*).

3.1. THEOREM. *Let $\mathbb{c}$ be a small $\mathcal{V}$-category; then $\mathfrak{Prof}_{\mathcal{V}}(\mathbb{c},\mathbb{c})$ is $\owedge$-semicyclic star-autonomous. Moreover, if $\mathcal{V}$ is also quasicyclic (and therefore cyclic), then the same is true of $\mathfrak{Prof}_{\mathcal{V}}(\mathbb{c},\mathbb{c})$.*

Before proceeding with the proof, we discuss a few of the issues that arise in the consideration of non-symmetric $\mathcal{V}$.

3.2. REMARK. Given an arbitrary ($\owedge$-semi)cyclic star-autonomous category $\mathcal{V}$, it is impossible to define the product $\mathbb{a} \otimes \mathbb{b}$ of $\mathcal{V}$-categories $\mathbb{a}$ and $\mathbb{b}$; this requires at least a braiding on $\mathcal{V}$. Similarly, it is impossible to define the opposite $\mathbb{c}^{\mathsf{op}}$ of a $\mathcal{V}$-category $\mathbb{c}$; this requires at least a braiding or an *involution* in the sense of [Egg10].

Thus, the notion of $\mathcal{V}$-profunctor $\mathbb{a} \nrightarrow \mathbb{b}$ must be defined in more elementary terms than the customary "$\mathcal{V}$-functor $\mathbb{a}^{\mathsf{op}} \otimes \mathbb{b} \longrightarrow \mathcal{V}$". This is done in [Bén73], and also in [Str83]: very simply, a $\mathcal{V}$-profunctor $f : \mathbb{a} \nrightarrow \mathbb{b}$ is an $(\mathsf{ob}\,\mathbb{a} \times \mathsf{ob}\,\mathbb{b})$-indexed family of $\mathcal{V}$-objects, $\langle q, r\rangle_f$, together with $\mathcal{V}$-arrows

$$\langle p,q\rangle_{\mathbb{a}} \owedge \langle q,r\rangle_f \xrightarrow{\langle p,q,r\rangle_;} \langle p,r\rangle_f \qquad\qquad \langle q,r\rangle_f \owedge \langle r,s\rangle_{\mathbb{b}} \xrightarrow{\langle q,r,s\rangle_;} \langle q,s\rangle_f$$

(for all $p, q \in \mathsf{ob}\,\mathbb{a}$ and $r, s \in \mathsf{ob}\,\mathbb{b}$) satisfying the five obvious associativity and unitality axioms.

[It is perhaps helpful to imagine $\langle q, r\rangle_f$ as consisting of *oblique* arrows $q \longrightarrow r$, and $\langle p, q, r\rangle_;$ and $\langle q, r, s\rangle_;$ as performing the composition of these with *genuine* arrows $p \longrightarrow q$ and $r \longrightarrow s$ in $\mathbb{a}$ and $\mathbb{b}$ respectively. For example, the identity profunctor on $\mathbb{c}$ is obtained by regarding (the object of) genuine arrows in $\mathbb{c}$ as (an object of) oblique arrows.]

A *modulation* of $\mathcal{V}$-profunctors $\omega : f \Rightarrow g$ is a family of $\mathcal{V}$-arrows $\langle q, r\rangle_\omega : \langle q, r\rangle_f \longrightarrow \langle q, r\rangle_g$ which are suitably compatible with the multiplicative structure of f and g. (We borrow the term modulation from [CKSW03].)

Composition of $\mathcal{V}$-profunctors is a routine application of *coends*: given profunctors $f : \mathbb{a} \nrightarrow \mathbb{b}$ and $g : \mathbb{b} \nrightarrow \mathbb{c}$, we take the family

$$\langle q, s\rangle_{f \owedge g} := \int^r \langle q, r\rangle_f \owedge \langle r, s\rangle_g,$$

together with a left $\mathbb{a}$-action derived from that of f, and a right $\mathbb{c}$-action derived from that of g; see [Bén73] or [Str83] for details.

Using these more elementary definitions, it is not clear that $\mathfrak{Prof}_{\mathcal{V}}(\mathbb{c},\mathbb{c})$ should be closed; indeed, it appears to us to be untrue in full generality. Certainly, we have (so far) been unable to deduce a star-autonomous structure on $\mathfrak{Prof}_{\mathcal{V}}(\mathbb{c},\mathbb{c})$ when $\mathcal{V}$ is an arbitrary star-autonomous category: it seems that the hypothesis of $\owedge$-semicyclicity cannot be weakened further. This asymmetry (that $\owedge$-semicyclicity is essential and $\ovee$-semicyclicity optional) stems partly from the asymmetry contained in the very definitions of $\mathcal{V}$-category and $\mathcal{V}$-profunctor (which are cast in terms of $\owedge$ and not $\ovee$), and partly from the inherent "two-dimensionality" of the notion of $\mathcal{V}$-matrix, which underlies that of $\mathcal{V}$-profunctor.

Proof. Let $\mathbb{a}$ and $\mathbb{b}$ be small $\mathcal{V}$-categories, and f a $\mathcal{V}$-profunctor $\mathbb{a} \nrightarrow \mathbb{b}$. Then we use the "contraposition" isomorphisms $x \multimap y \longrightarrow x^* \multimapinv y^*$ and $z \multimapinv x \longrightarrow {}^*z \multimap {}^*x$ in $\mathcal{V}$ to construct $\mathcal{V}$-arrows

$$\frac{\dfrac{\dfrac{\langle r,p\rangle_f \owedge \langle p,q\rangle_{\mathbb{b}} \longrightarrow \langle r,q\rangle_f}{\langle p,q\rangle_{\mathbb{b}} \longrightarrow \langle r,p\rangle_f \multimap \langle r,q\rangle_f}}{\langle p,q\rangle_{\mathbb{b}} \longrightarrow (\langle r,p\rangle_f)^* \multimapinv (\langle r,q\rangle_f)^*}}{\langle p,q\rangle_{\mathbb{b}} \owedge (\langle r,q\rangle_f)^* \longrightarrow (\langle r,p\rangle_f)^*} \qquad \frac{\dfrac{\dfrac{\langle r,s\rangle_{\mathbb{a}} \owedge \langle s,q\rangle_f \longrightarrow \langle r,q\rangle_f}{\langle r,s\rangle_{\mathbb{a}} \longrightarrow \langle r,q\rangle_f \multimapinv \langle s,q\rangle_f}}{\langle r,s\rangle_{\mathbb{a}} \longrightarrow {}^*(\langle r,q\rangle_f) \multimap {}^*(\langle s,q\rangle_f)}}{{}^*(\langle r,q\rangle_f) \owedge \langle r,s\rangle_{\mathbb{a}} \longrightarrow {}^*(\langle s,q\rangle_f)}$$

which exhibit: a left action of $\mathbb{b}$ on the family of objects $\langle q,r\rangle_{f^*} := (\langle r,q\rangle_f)^*$; and, a right action of $\mathbb{a}$ on the family of objects $\langle q,r\rangle_{{}^*f} := {}^*(\langle r,q\rangle_f)$. In other words, we obtain profunctors $f^* : \mathbb{b} \nrightarrow \mathbb{1}$ and ${}^*f : \mathbb{1} \nrightarrow \mathbb{a}$.

At this point it is natural to use φ to transport the left action of $\mathbb{b}$ on f^* to one on ${}^*f$ and to transport the right action of $\mathbb{a}$ on ${}^*f$ to one on f^*, as follows:

$$\langle p,q\rangle_{\mathbb{b}} \owedge {}^*(\langle r,q\rangle_f) \xrightarrow{\mathrm{id} \owedge \varphi^{-1}} \langle p,q\rangle_{\mathbb{b}} \owedge (\langle r,q\rangle_f)^* \longrightarrow (\langle r,p\rangle_f)^* \xrightarrow{\varphi} {}^*(\langle r,p\rangle_f)$$

$$(\langle r,q\rangle_f)^* \owedge \langle r,s\rangle_{\mathbb{a}} \xrightarrow{\varphi \owedge \mathrm{id}} {}^*(\langle r,q\rangle_f) \owedge \langle r,s\rangle_{\mathbb{a}} \longrightarrow {}^*(\langle s,q\rangle_f) \xrightarrow{\varphi^{-1}} (\langle s,q\rangle_f)^*$$

To show that we obtain profunctors $f^* : \mathbb{b} \nrightarrow \mathbb{a}$ and ${}^*f : \mathbb{b} \nrightarrow \mathbb{a}$ from the four actions described above, it remains to show that middle associativity holds; this is surprisingly difficult.

An alternative approach, favoured by the second author, is to derive maps $\langle p,q\rangle_{\mathbb{b}} \owedge {}^*(\langle r,q\rangle_f) \longrightarrow {}^*(\langle r,p\rangle_f)$ and $(\langle r,q\rangle_f)^* \owedge \langle r,s\rangle_{\mathbb{a}} \longrightarrow (\langle s,q\rangle_f)^*$ by deCurrying each of the following composites.

$$\begin{tikzcd}
 & {}^*\langle p,q\rangle_{\mathbb{b}} \ovee {}^*(\langle r,p\rangle_f) \arrow[r, "\varphi \ovee \mathrm{id}"] & \langle p,q\rangle_{\mathbb{b}}{}^* \ovee {}^*(\langle r,p\rangle_f) \arrow[d, dashed, "\wr"] \\
{}^*(\langle r,q\rangle_f) \arrow[r] & {}^*(\langle r,p\rangle_f \owedge \langle p,q\rangle_{\mathbb{b}}) \arrow[u, dashed, "\wr"] & \langle p,q\rangle_{\mathbb{b}} \multimap {}^*(\langle r,p\rangle_f)
\end{tikzcd}$$

$$\begin{tikzcd}
(\langle r,q\rangle_f)^* \arrow[r] & (\langle r,s\rangle_{\mathbb{a}} \owedge \langle s,q\rangle_f)^* \arrow[d, dashed, "\wr"] & (\langle s,q\rangle_f)^* \multimapinv \langle r,s\rangle_{\mathbb{a}} \\
 & (\langle s,q\rangle_f)^* \ovee \langle r,s\rangle_{\mathbb{a}}{}^* \arrow[r, "\varphi \ovee \mathrm{id}"] & (\langle s,q\rangle_f)^* \ovee {}^*\langle r,s\rangle_{\mathbb{a}} \arrow[u, dashed, "\wr"]
\end{tikzcd}$$

The advantage of this approach is that middle associativity becomes trivial; the disadvantage is that ordinary (left- and right-) associativity becomes difficult.

The solution is to show that both approaches result in the same pair of arrows. This is a simple exercise, yet it relies crucially on axiom (T_2): see Lemma A.3.

By contrast, it is essentially tautologous that the family of $\mathcal{V}$-arrows

$$\langle q,r\rangle_{f^*} = (\langle r,q\rangle_f)^* \xrightarrow{\varphi_{\langle r,q\rangle_f}} {}^*(\langle r,q\rangle_f) = \langle q,r\rangle_{{}^*f}$$

defines an invertible modulation of profunctors $\widetilde{\varphi}_f : f^* \longrightarrow {}^*f$.

Now we define $\ovee$ and d as the de Morgan duals of $\owedge$ and e; it is easy to work out that $f \ovee g$ can be equivalently defined using *ends*:

$$\langle q,s\rangle_{f \ovee g} := \int_r \langle q,r\rangle_f \ovee \langle r,s\rangle_g,$$

together with a left $\mathbb{c}$-action derived from that of f, and a right $\mathbb{c}$-action derived from that of g. Constructing the necessary linear distribution is routine: a modulation $f \owedge (g \ovee h) \longrightarrow (f \owedge \; g) \ovee h$ is uniquely determined by an appropriate family of arrows $\langle p,q\rangle_f \owedge \langle q,s\rangle_{g \ovee h} \longrightarrow \langle p,r\rangle_{f \owedge g} \ovee \langle r,s\rangle_h$, and it is neither hard to see that

$$\begin{tikzcd}
\langle p,q\rangle_f \owedge \langle q,s\rangle_{g \ovee h} \arrow[d, "\mathrm{id} \owedge \pi_r"'] & \langle p,r\rangle_{f \owedge g} \ovee \langle r,s\rangle_h \\
\langle p,q\rangle_f \owedge (\langle q,r\rangle_g \ovee \langle r,s\rangle_h) \arrow[r, "\vec{\kappa}"] & (\langle p,q\rangle_f \owedge \langle q,r\rangle_g) \ovee \langle r,s\rangle_h \arrow[u, "\upsilon_q \ovee \mathrm{id}"']
\end{tikzcd}$$

is such a family of arrows, nor that the resultant modulations satisfy the necessary coherence axioms (compare with [Egg08]). Thus $\mathfrak{Prof}_{\mathcal{V}}$ is a linear bicategory; in particular, $\mathfrak{Prof}_{\mathcal{V}}(\mathbb{c},\mathbb{c})$ is a linearly distributive category for every $\mathbb{c}$.

The construction of modulations

$$e \xrightarrow{\tau} f^* \ovee f, \quad e \xrightarrow{\tau} f \ovee {}^*f, \quad f \owedge f^* \xrightarrow{\gamma} d \quad \text{and} \quad {}^*f \owedge f \xrightarrow{\gamma} d$$

satisfying the necessary (linear) triangle identities (thus proving f^* and ${}^*f$ to be, respectively, right and left duals of f) is similarly routine. Thus $\mathfrak{Prof}_{\mathcal{V}}$ is a $*$-linear bicategory; in particular, $\mathfrak{Prof}_{\mathcal{V}}(\mathbb{c},\mathbb{c})$ is a star-autonomous category.

Given a modulation $\omega : f \longrightarrow g$, the dual modulation $\omega^* : g^* \longrightarrow f^*$ is calculated pointwise, as one would expect.

$$\begin{tikzcd}
\langle p,q\rangle_{g^*} \arrow[r, "\langle p,q\rangle_{\omega^*}"] \arrow[d, equal] & \langle p,q\rangle_{f^*} \arrow[d, equal] \\
(\langle q,p\rangle_g)^* \arrow[r, "(\langle q,p\rangle_\omega)^*"'] & (\langle q,p\rangle_f{}^*)
\end{tikzcd}$$

Moreover, the cancellation modulations $f \longrightarrow ({}^*f)^*$ and $f \longrightarrow ({}^*f)^*$, can also be calculated pointwise; it follows that the de Morgan modulations, such as $(f \owedge g)^* \longrightarrow g^* \ovee f^*$, are related to those of $\mathcal{V}$ as follows.

$$\begin{tikzcd}
\langle p,r\rangle_{f \owedge g^*} \arrow[r, "\vartheta"] \arrow[d, equal] & \langle p,r\rangle_{g^* \ovee f^*} \arrow[r, "\pi_q"] & \langle p,q\rangle_{g^*} \ovee \langle q,r\rangle_{f^*} \arrow[d, equal] \\
(\langle r,p\rangle_{f \owedge g})^* \arrow[r, "\upsilon_q{}^*"'] & (\langle r,q\rangle_f \owedge \langle q,p\rangle_g)^* \arrow[r, "\vartheta"'] & (\langle q,p\rangle_g)^* \ovee (\langle r,q\rangle_f{}^*)
\end{tikzcd}$$

These observations allow one to quickly conclude that the $\owedge$-semicyclicity of $\mathcal{V}$ is inherited by $\mathfrak{Prof}_{\mathcal{V}}(\mathbb{c},\mathbb{c})$, and also the quasicyclicity, if $\mathcal{V}$ enjoys that property. ∎

4. Braidings and cyclicities

Throughout this section: $\mathcal{K} = (\mathcal{K}, \owedge, e, \ovee, d, (\ )^*, {}^*(\ ))$ denotes a braided star-autonomous category; ζ, a natural isomorphism of $\mathrm{Id}_\mathcal{K} \longrightarrow \mathrm{Id}_\mathcal{K}$; φ, a natural isomorphism $(\ )^* \longrightarrow {}^*(\ )$; and Φ the corresponding natural isomorphism $\langle - \owedge ?, d\rangle_\mathcal{K} \longrightarrow \langle ? \owedge -, d\rangle_\mathcal{K}$.

4.1. Remark. By a *braided star-autonomous category* we mean simply a star-autonomous category together with a braiding for $\owedge$. By duality, this automatically induces a braiding for $\ovee$—hence, there are actually two braidings: we write $\hat{\beta}$ for the braiding on $\owedge$, and $\check{\beta}$ for the braiding on $\ovee$.

These two braidings cohere with one another in the sense that the four diagrams

$$\begin{tikzcd}[column sep=tiny]
 & (p \ovee q) \owedge r \arrow[dl, "\check{\beta}^{-1}_{p,q} \owedge \mathrm{id}_r"'] \arrow[dr, "\hat{\beta}_{p \ovee q, r}"] & \\
(q \ovee p) \owedge r \arrow[ddd, "\overleftarrow{\kappa}_{q,p,r}"'] & & r \owedge (p \ovee q) \arrow[ddd, "\overrightarrow{\kappa}_{r,p,q}"] \\
 & r \owedge (q \ovee p) \arrow[ul, dotted, "\hat{\beta}^{-1}_{q \ovee p, r}"] \arrow[ur, dotted, "\mathrm{id}_r \owedge \check{\beta}_{p,q}"'] & \\
 & q \ovee (r \owedge p) & \\
q \ovee (p \owedge r) \arrow[ur, dotted, "\mathrm{id}_q \ovee \hat{\beta}_{r,p}"] \arrow[dr, "\check{\beta}_{q, p \owedge r}"'] & & (r \owedge p) \ovee q \arrow[ul, dotted, "\check{\beta}^{-1}_{q, r \owedge p}"'] \arrow[dl, "\hat{\beta}^{-1}_{r,p} \ovee \mathrm{id}_q"] \\
 & (p \owedge r) \ovee q &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
 & r \owedge (q \ovee p) \arrow[dl, "\hat{\beta}_{q \ovee p, r}"'] \arrow[dr, "\mathrm{id}_r \owedge \check{\beta}^{-1}_{p,q}"] & \\
(q \ovee p) \owedge r \arrow[ddd, "\overleftarrow{\kappa}_{q,p,r}"'] & & r \owedge (p \ovee q) \arrow[ddd, "\overrightarrow{\kappa}_{r,p,q}"] \\
 & (p \ovee q) \owedge r \arrow[ul, dotted, "\check{\beta}_{p,q} \owedge \mathrm{id}_r"] \arrow[ur, dotted, "\hat{\beta}^{-1}_{p \ovee q, r}"'] & \\
 & (p \owedge r) \ovee q & \\
q \ovee (p \owedge r) \arrow[ur, dotted, "\check{\beta}^{-1}_{q, p \owedge r}"] \arrow[dr, "\mathrm{id}_q \ovee \hat{\beta}^{-1}_{r,p}"'] & & (r \owedge p) \ovee q \arrow[ul, dotted, "\hat{\beta}_{r,p} \ovee \mathrm{id}_q"'] \arrow[dl, "\check{\beta}_{q, r \owedge p}"] \\
 & q \ovee (r \owedge p) &
\end{tikzcd}$$

hold (see Lemma A.4). (Compare with the definition of symmetric star-autonomous category in [CS97, §3].) This means that the non-standard wire-crossings listed in (the central column of) Figure 3 are well-defined. It also entails that in the degenerate case where $\owedge = \ovee$, $e = d$, $\overleftarrow{\kappa} = \alpha$ and $\overrightarrow{\kappa} = \alpha^{-1}$, one can derive $\hat{\beta} = \check{\beta}$; simply set $p = e = d$ in the first solid diagram above, and reduce appropriately.

In light of the remarks above, it will be convenient to use the following (provisional) terminology.

4.2. Definitions. *Let ζ be a natural isomorphism* $\mathrm{Id}_\mathcal{K} \longrightarrow \mathrm{Id}_\mathcal{K}$*. Then we call ζ: a* $\owedge$-semibalance for $\mathcal{K}$ *if it is a balance for $\hat{\beta}$—that is, if* ($\hat{\mathrm{B}}$) *holds; a* $\ovee$-semibalance for $\mathcal{K}$ *if it is a balance for $\check{\beta}$—that is, if* ($\check{\mathrm{B}}$) *holds; a* balance for $\mathcal{K}$ *if it is both a $\owedge$-semibalance*

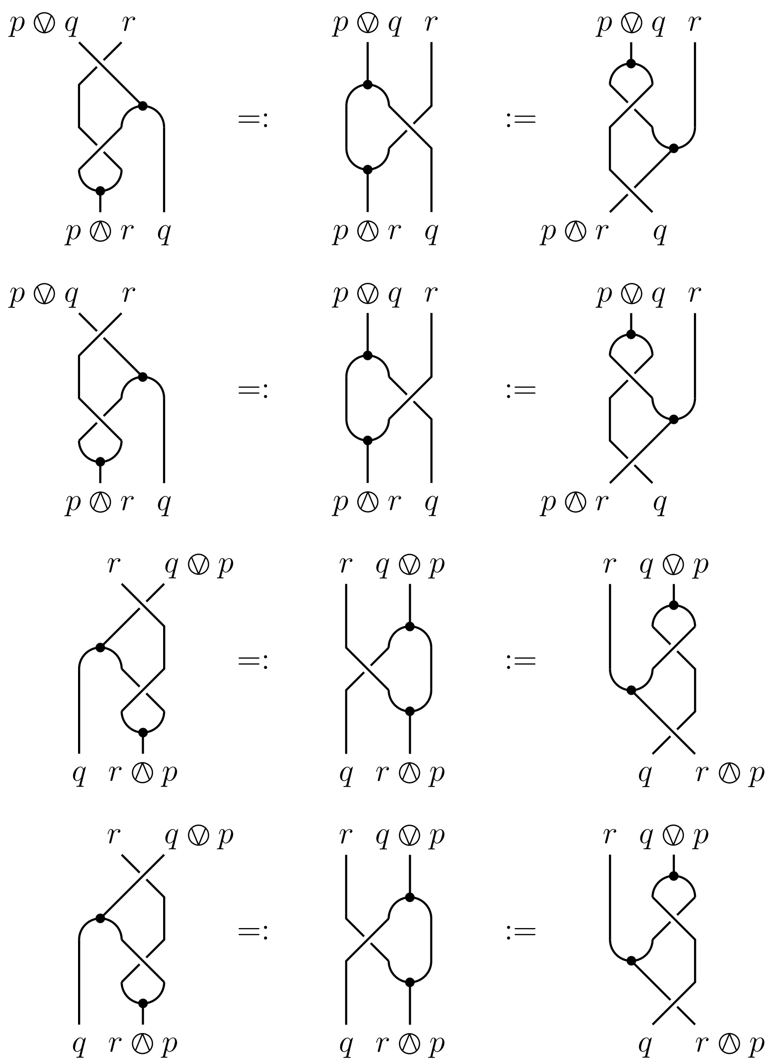


Figure 3: Non-planar linear distributions in a braided star-autonomous category

and a $\ovee$-semibalance—that is, if both ($\hat{\mathrm{B}}$) *and* ($\check{\mathrm{B}}$) *hold.*

$$
\begin{tikzcd}
p \owedge q \arrow[r, "\hat{\beta}_{p,q}"] \arrow[d, "\zeta_{p \owedge q}"'] & q \owedge p \arrow[d, "\zeta_q \owedge \zeta_p"] \arrow[dl, phantom, "(\hat{\mathrm{B}})"] \\
p \owedge q & q \owedge p \arrow[l, "\hat{\beta}_{q,p}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
p \ovee q \arrow[r, "\hat{\beta}_{p,q}"] \arrow[d, "\zeta_{p \ovee q}"'] & q \ovee p \arrow[d, "\zeta_q \ovee \zeta_p"] \arrow[dl, phantom, "(\check{\mathrm{B}})"] \\
p \ovee q & q \ovee p \arrow[l, "\hat{\beta}_{q,p}"]
\end{tikzcd}
$$

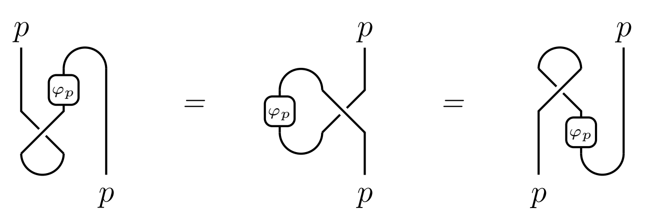
Note that ($\hat{\mathrm{B}}$) entails $\zeta_e = \mathrm{id}_e$ and that ($\check{\mathrm{B}}$) entails $\zeta_d = \mathrm{id}_d$ for the same reason as (T_2) $\Rightarrow$ (T_0) and (P_2) $\Rightarrow$ (P_0)—see, again, Lemma A.1.

4.3. Theorem. *There are bijective correspondences between: $\owedge$-semicycles and $\owedge$-semibalances; $\ovee$-semicycles and $\ovee$-semibalances; cycles and balances.*

This theorem will be proven by series of lemmata following the graphical intuitions laid out in section 1. (But, since our ambient star-autonomous category is only assumed to be braided, we use wire/strings instead of tape/ribbons.)

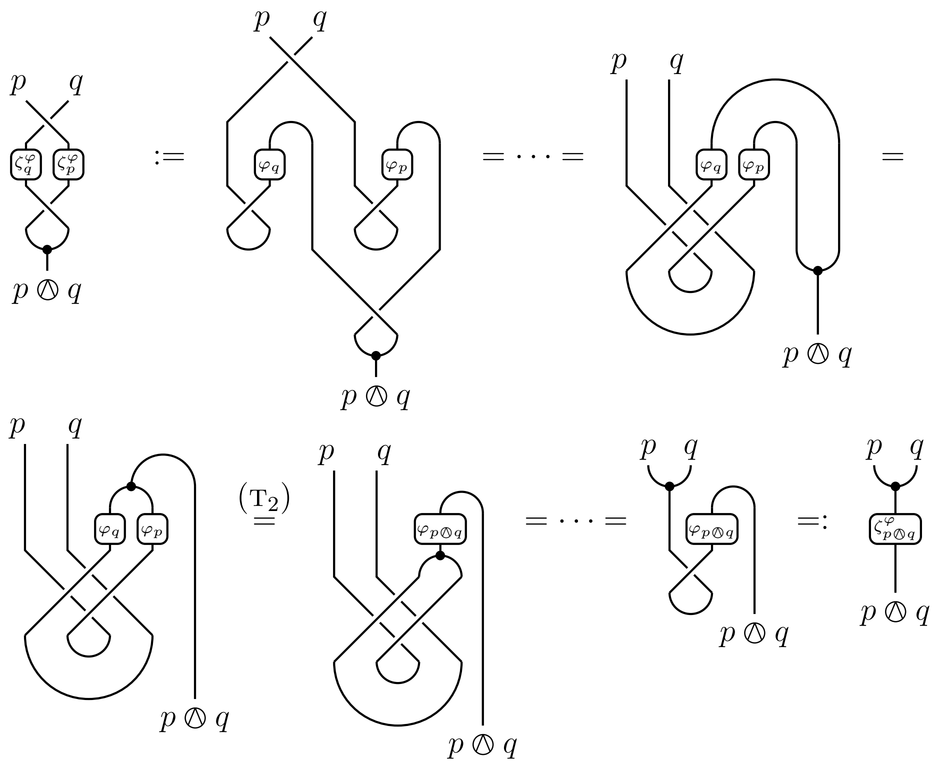
4.4. Lemma. *Let ζ^φ be the natural transformation whose components are given by (the common composite of) the diagram below—or, equivalently, by (the common value of) the string diagrams which follow.*

$$
\begin{tikzcd}
p \arrow[r, "\rho_p^{-1}"] \arrow[dr, bend left, "\lambda_p^{-1}"'] & p \owedge e \arrow[r, "\mathrm{id}_p \owedge \tau"] & p \owedge (p^* \ovee p) \arrow[r, "\vec{\kappa}_{p,p^*,p}(\varphi_p)"] & (p \owedge {}^*p) \ovee p \arrow[d, dashed, "\hat{\beta}^{-1}_{p,{}^*p} \ovee \mathrm{id}_p"'] & & \\
& e \owedge p \arrow[u, dashed, "\hat{\beta}_{e,p}"] \arrow[r, "\tau \owedge \mathrm{id}_p"] & (p^* \ovee p) \owedge p \arrow[u, dashed, "\hat{\beta}_{p^* \ovee p,p}"] \arrow[r, dotted] \arrow[d, dashed, "\check{\beta}^{-1}_{p^*,p} \owedge \mathrm{id}_p"'] & ({}^*p \owedge p) \ovee p \arrow[r, "\gamma \ovee \mathrm{id}_p"] & d \ovee p \arrow[d, dashed, "\check{\beta}_{d,p}"'] \arrow[dr, bend left, "\lambda_p"] & \\
& & (p \ovee p^*) \owedge p \arrow[r, "\overleftarrow{\kappa}_{p,p^*,p}(\varphi_p)"'] & p \ovee ({}^*p \owedge p) \arrow[u, dashed, "\check{\beta}_{p,{}^*p \owedge p}"] \arrow[r, "\mathrm{id}_p \ovee \gamma"'] & p \ovee d \arrow[r, "\rho_p"'] & p
\end{tikzcd}
$$



If φ is a $\owedge$-semicycle, then ζ^φ is a $\owedge$-semibalance. Similarly, if φ is a $\ovee$-semicycle, then ζ^φ is a $\ovee$-semibalance. Hence, if φ is a cycle, then ζ^φ is a balance.

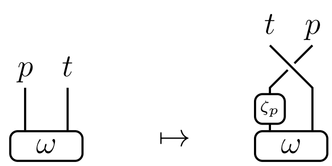
Proof. We give a graphical proof of (T_2) $\Rightarrow$ ($\hat{\mathrm{B}}$):



The proof of (P_2) $\Rightarrow$ ($\check{\mathrm{B}}$) is exactly dual. ∎

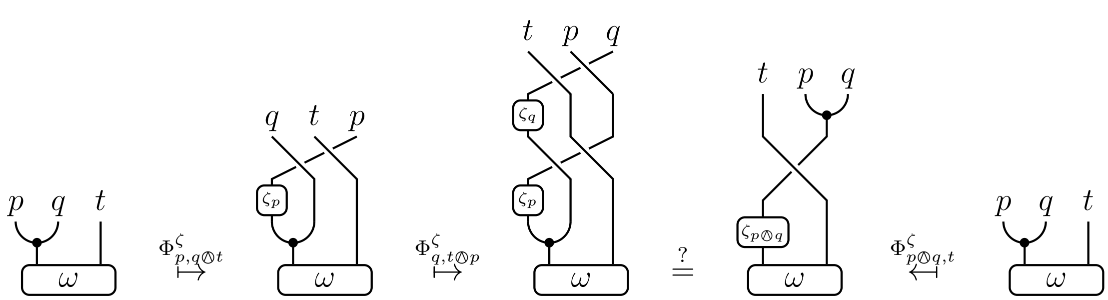
4.5. **Lemma.** *Let Φ^ζ be the operation on external hom-sets given below.*

$$\Phi^{\zeta}_{p,t}(\omega) := \beta_{p,t} \,;\, (\zeta_p \owedge \mathrm{id}_t) \,;\, \omega$$

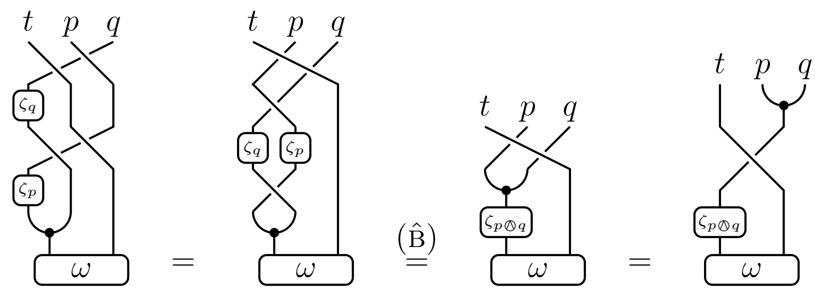


If ζ is a $\owedge$-semibalance, then Φ^ζ is a $\owedge$-semicycle. Similarly, if ζ is a $\ovee$-semibalance, then Φ^ζ is a $\ovee$-semicycle. Hence, if ζ is a balance, then Φ^ζ is a cycle.

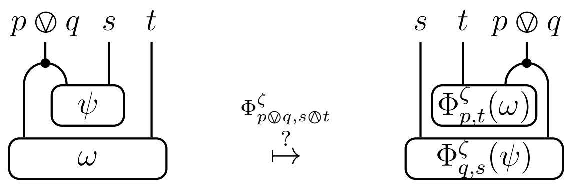
Proof. To prove ($\hat{\mathrm{B}}$) $\Rightarrow$ (E), we need to show that the following equality holds, for every arrow $(p \owedge q) \owedge t \xrightarrow{\omega} d$.



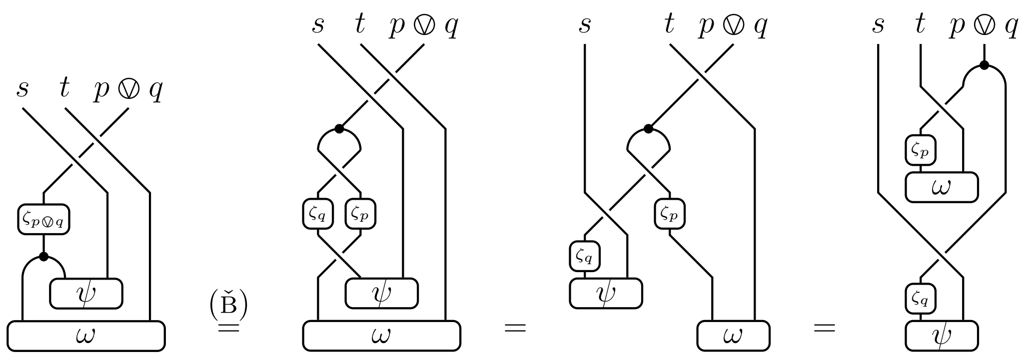
This is easily established by the argument below.



To prove ($\check{\text{B}}$) $\Rightarrow$ (M_2), we need to show that, for every pair of arrows $p \oslash t \xrightarrow{\omega} d$ and $q \oslash s \xrightarrow{\psi} d$, the result of chasing (ω, ψ) along the lower path of (M_2) equals the result of chasing it along the upper path.



Again, this is easily proven, as follows.

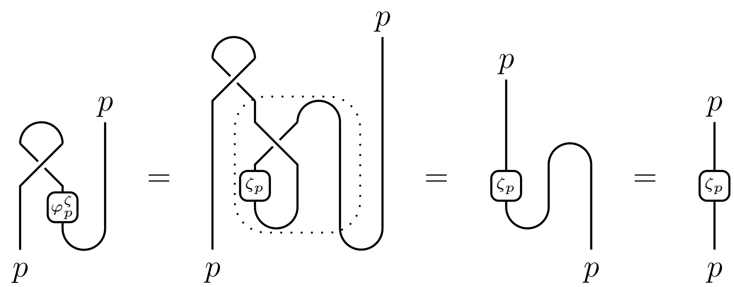


■

4.6. Lemma. *The two constructions outlined above are inverse to one another—that is, $\Phi^{\zeta^\varphi} = \Phi$ and $\zeta^{\varphi^\zeta} = \zeta$ (where φ^ζ corresponds to Φ^ζ).*

Proof. That, for any given φ, $\Phi^{\zeta^\varphi} = \Phi$ is simply a more rigorous version of the argument appearing in section 1: one simply substitutes a $=:$ for a $\overset{?}{=}$.

The converse requires a little more work: given a ζ, we must work out the φ^ζ corresponding to Φ^ζ. According to Remark 1.3, we have $\varphi^\zeta_p = \mathsf{rCurry}(\Phi^\zeta_{p,p^*}(\mathsf{lCurry}^{-1}(\mathrm{id}_{p^*})))$. Hence, $\zeta_p^{\varphi^\zeta}$ equals

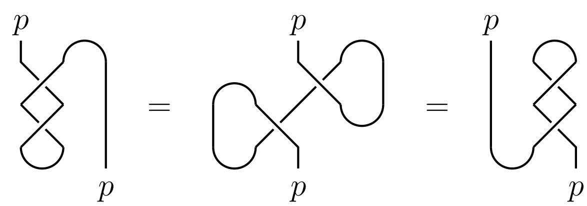


This concludes the proof of Theorem 4.3. ■

By similar arguments, it is possible to show, for arbitrary natural isomorphisms $\zeta : \mathrm{Id}_{\mathcal{K}} \longrightarrow \mathrm{Id}_{\mathcal{K}}$, that $\zeta_e = \mathrm{id}_e$ if and only if φ^ζ satisfies (T_0), and that $\zeta_d = \mathrm{id}_d$ if and only if φ^ζ satisfies (P_0). Applying Lemma 2.3, one sees that ($\hat{\mathrm{B}}$) and $\zeta_d = \mathrm{id}_d$ (or, alternatively, ($\check{\mathrm{B}}$) and $\zeta_e = \mathrm{id}_e$) suffice to show that ζ is a balance for $\mathcal{K}$ in the sense of Definition 4.2.

4.7. Corollary. *Φ^{id} is a cycle if and only if $\hat{\beta}$ (and therefore also $\check{\beta}$) is a symmetry.*

We now turn to the question of quasicyclicity; in particular, whether it is possible that Φ^{id} be a quasicycle even if $\hat{\beta}$ is not a symmetry. Observe that every object p of a braided star-autonomous category $\mathcal{K}$ admits a canonical 4π-*twist*:



—we shall denote this map ξ_p. Of course, if the braiding happens to be a symmetry, then ξ_p will be the identity for all p. The converse is false: Gabriella Böhm and the second author have together constructed a class of braided Hopf algebras H with the property that $\mathbf{mod}_H$, the category of finite-dimensional H-modules, satisfies $\xi = \mathrm{id}$ and $\beta^2 \neq \mathrm{id}$. The simplest of these is the Drinfeld double of (the group algebra of) $\mathbb{Z}_2$ together with its universal R-matrix.

4.8. Theorem. *Using the same notation as before, Φ^ζ is a quasicycle if and only if*

$$p \xrightarrow{\quad \zeta_p \quad} p \xrightarrow{\quad \eta \quad} {}^*(p^*) \xrightarrow{\quad {}^*(\zeta_{p^*}) \quad} {}^*(p^*) \xrightarrow{\quad \eta^{-1} \quad} p$$

equals ξ_p for all p. In particular, Φ^{id} is a quasicycle if and only if $\xi_p = \mathrm{id}_p$ for all p.

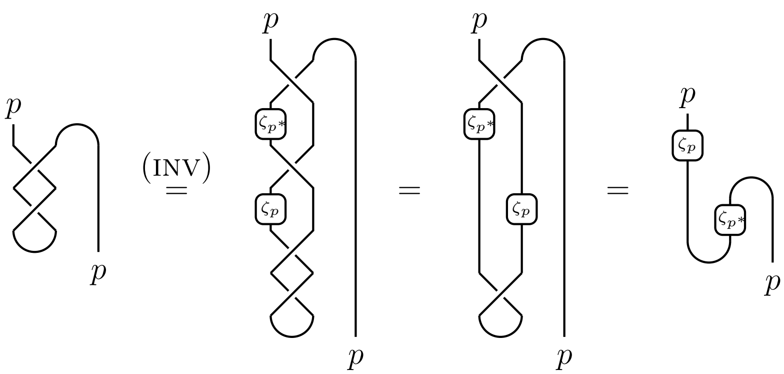
Naturally, we shall call a natural transformation $\mathrm{Id}_{\mathcal{K}} \xrightarrow{\zeta} \mathrm{Id}_{\mathcal{K}}$ a *quasibalance* if it satisfies this condition. (An equivalent and arguably more elegant condition is that $\xi_{p^*} =$

$(\xi_p)^*$ should equal

$$\begin{tikzcd}
p^* & p^* \arrow[l, "(\zeta_p)^*"'] & ({}^*(p^*))^* \arrow[l, "\eta^*"'] & ({}^*(p^*))^* \arrow[l, "({}^*(\zeta_{p^*}))^*"'] & p^* \arrow[l, "(\eta^{-1})^*"'] \arrow[dl, bend left, "\mathrm{id}_{p^*}"] \\
 & & p^* \arrow[ul, bend left, "\mathrm{id}_{p^*}"] \arrow[u, "\eta"] & p^* \arrow[l, "\zeta_{p^*}"] \arrow[u, "\eta"'] &
\end{tikzcd}$$

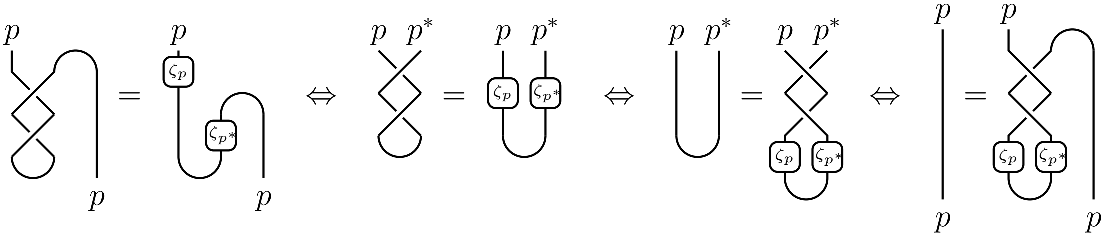
for all p.)

Proof of Theorem 4.8. Suppose that Φ^ζ is a quasicycle; then

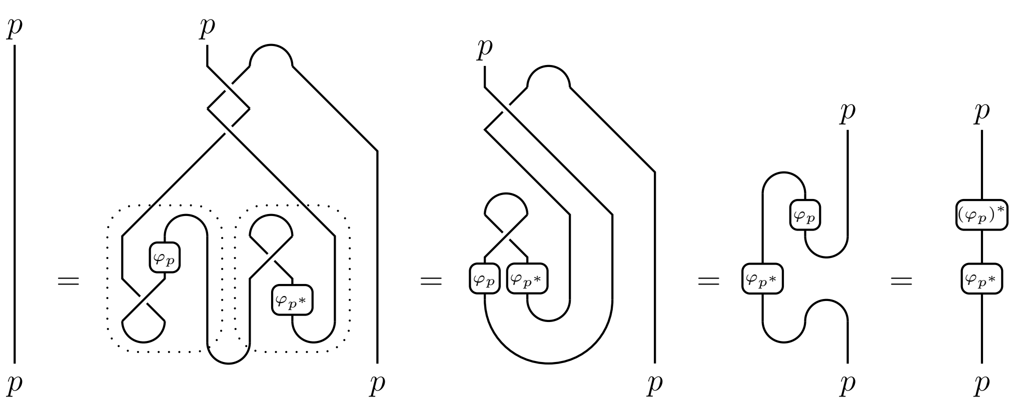


as desired.

Conversely, we note that the following are equivalent:



Now suppose that ζ^φ is a quasibalance; then



∎

Finally we note that, since every cycle is a quasicycle, every balance (on a braided star-autonomous category) is a quasibalance; from this one derives the following.

4.9. COROLLARY. *A balance ζ for a star-autonomous category $\mathcal{K}$ satisfies $\zeta_{p^*} = (\zeta_p)^*$ if and only if it satisfies $\xi_p = \zeta_p^2$.*

This very important property will be further discussed in [EM10].

5. Strictification

To conclude, we address the issue of strictifying negation in star-autonomous categories, in both cyclic and arbitrary cases. It seems otiose to rigorously state and prove theorems (which would entail, among other things, fully written-out definitions of *morphism* of cyclic star-autonomous categories, and of *two-cell* between such morphisms) when the truth of what we assert is so manifest. Consequently, we proceed in a slightly less formal fashion than heretofore; our techniques are simple and obvious extensions of those used in [CHS06].

5.1. DEFINITIONS. *Let $\mathcal{K} = (\mathcal{K}, \owedge, e, \ovee, d, (\ )^*, {}^*(\ ))$ be an arbitrary star-autonomous category; then, by a* $\mathbb{Z}$-string of (linear) adjoints *we mean a $\mathbb{Z}$-indexed family of $\mathcal{K}$-objects, $p = \{p_n\}_{n\in\mathbb{Z}}$, together with $\mathcal{K}$-arrows*

$$e \xrightarrow{\tau_n} p_{n+1} \ovee p_n \qquad\qquad p_n \owedge p_{n+1} \xrightarrow{\gamma_n} d$$

satisfying the (linear) triangle identities of [CS97]; and, by the canonical *$\mathbb{Z}$-string of adjoints determined by a $\mathcal{K}$-object p, we mean the family $\tilde{p} = \{\tilde{p}_n\}_{n\in\mathbb{Z}}$ given by*

$$\tilde{p}_n := \begin{cases} p^{*\cdots *} & \text{if } n \text{ is positive} \\ p & \text{if } n \text{ is zero} \\ {}^{*\cdots *}p & \text{if } n \text{ is negative} \end{cases}$$

together with the canonical (chosen) linear adjunctions between $\tilde{p}_n$ and $\tilde{p}_{n+1}$.

Similarly, by a $\mathbb{Z}$-string of (linear) mates *$p = \{p_n\}_{n\in\mathbb{Z}} \longrightarrow \{q_n\}_{n\in\mathbb{Z}} = q$ we mean a $\mathbb{Z}$-indexed family of $\mathcal{K}$-arrows $\omega = \{\omega_n\}_{n\in\mathbb{Z}}$ with $\omega_n \in \langle p_n, q_n\rangle_{\mathcal{K}}$ if n is even, and $\omega_n \in \langle q_n, p_n\rangle_{\mathcal{K}}$ if n is odd, all satisfying the (linear) mateship relations of [CS97]; and by the* canonical *$\mathbb{Z}$-string of mates determined by a $\mathcal{K}$-arrow ω, we mean the family $\tilde{\omega} = \{\tilde{\omega}_n\}_{n\in\mathbb{Z}}$ given by*

$$\tilde{\omega}_n := \begin{cases} \omega^{*\cdots *} & \text{if } n \text{ is positive} \\ \omega & \text{if } n \text{ is zero} \\ {}^{*\cdots *}\omega & \text{if } n \text{ is negative} \end{cases}$$

The category of all $\mathbb{Z}$-strings of adjoints, with $\mathbb{Z}$-strings of mates between them, will be denoted $\mathbf{Adj}_{\mathbb{Z}}(\mathcal{K})$.

It is evident that $\mathbf{Adj}_{\mathbb{Z}}(\mathcal{K})$ carries a star-autonomous structure, given (in part) by

$$(p \owedge q)_n = \begin{cases} p_n \owedge q_n & \text{if } n \text{ is even} \\ q_n \ovee p_n & \text{if } n \text{ is odd} \end{cases} \qquad (p \ovee q)_n = \begin{cases} p_n \ovee q_n & \text{if } n \text{ is even} \\ q_n \owedge p_n & \text{if } n \text{ is odd} \end{cases}$$

$$e_n = \begin{cases} e & \text{if } n \text{ is even} \\ d & \text{if } n \text{ is odd} \end{cases} \qquad d_n = \begin{cases} d & \text{if } n \text{ is even} \\ e & \text{if } n \text{ is odd} \end{cases}$$

$$(p^*)_n = p_{n+1} \qquad ({}^*p)_n = p_{n-1}$$

and that this has *strict negations* in the sense described in section 1. Moreover, $\tilde{(\ )}$ and $(\ )_0$ define an (adjoint) equivalence of star-autonomous categories between $\mathcal{K}$ and $\mathbf{Adj}_{\mathbb{Z}}(\mathcal{K})$. Hence every star-autonomous category is equivalent to one with strict negations. It follows that any cycle φ on $\mathcal{K}$ can be extended to a cycle $\tilde{\varphi}$ on $\mathbf{Adj}_{\mathbb{Z}}(\mathcal{K})$; explicitly,

$$(p^*)_n = p_{n+1} \xrightarrow{\quad\sim\quad} (p_n)^* \xrightarrow{\quad\varphi_{p_n}\quad} {}^*(p_n) \xrightarrow{\quad\sim\quad} p_{n-1} = ({}^*p)_n$$

is the nth component of $\tilde{\varphi}_p$.

(Note also that the two monoidal structures of $\mathbf{Adj}_{\mathbb{Z}}(\mathcal{K})$ are strict if and only if the same is true of $\mathcal{K}$. Hence, to produce a fully strict star-autonomous category equivalent to $\mathcal{K}$, one could first strictify its linearly distributive structure, and then apply the $\mathbf{Adj}_{\mathbb{Z}}(-)$ construction.)

5.2. Definition. *Let $(\mathcal{K}, \varphi)$ be a cyclic star-autonomous category; then, by a* $\mathbb{Z}_2$-string of (linear) adjoints, *we mean a $\mathbb{Z}$-string of adjoints $p = \{p_n\}_{n\in\mathbb{Z}}$ satisfying*

$$p_{n+1} = p_{n-1} \qquad \textit{and} \qquad \gamma_n = \Phi_{p_{n-1},p_n}(\gamma_{n-1})$$

for all $n \in \mathbb{Z}$. The full subcategory of $\mathbf{Adj}_{\mathbb{Z}}(\mathcal{K})$ determined by the $\mathbb{Z}_2$-strings will be denoted $\mathbf{Adj}_{\mathbb{Z}_2}(\mathcal{K})$.

5.3. Lemma. *$\mathbf{Adj}_{\mathbb{Z}_2}(\mathcal{K})$ is a sub-star-autonomous category of $\mathbf{Adj}_{\mathbb{Z}}(\mathcal{K})$—that is, the class of $\mathbb{Z}_2$-strings is closed under $\owedge$, $\ovee$, e, d, $(\ )^*$ and ${}^*(\ )$. Moreover, the restriction of $\tilde{\varphi}$ to $\mathbf{Adj}_{\mathbb{Z}_2}(\mathcal{K})$ is the identity.*

Proof. For the first statement, we prove only that if p and q are $\mathbb{Z}_2$-strings, then so is $p \owedge q$. It is trivial that $(p \owedge q)_{n+1} = (p \owedge q)_{n-1}$, but it is non-trivial that the other condition still holds. The full definition of $p \owedge q$ (which was only partially described above) includes the following.

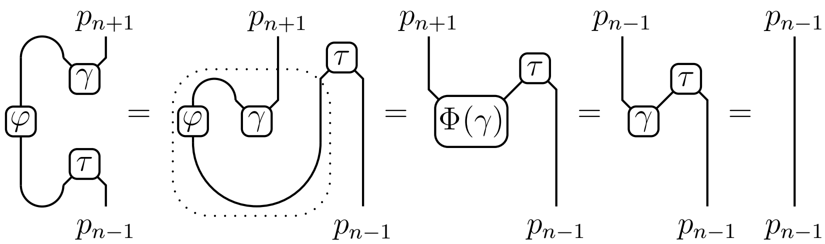
$$\begin{tikzcd}
(p \owedge q)_{2m-1} \owedge (p \owedge q)_{2m} \arrow[d, "\gamma_{2m-1}"', ":="] \arrow[dd, bend left, "\mathrm{id}"] \\
d \\
(q_{2m-1} \ovee p_{2m-1}) \owedge (p_{2m} \owedge q_{2m}) \arrow[u, "\mathsf{lbind}(\gamma_{2m-1}, \gamma_{2m-1})"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(p \owedge q)_{2m} \owedge (p \owedge q)_{2m+1} \arrow[d, "\gamma_{2m}"', ":="] \arrow[dd, bend left, "\mathrm{id}"] \\
d \\
(p_{2m} \owedge q_{2m}) \owedge (q_{2m+1} \ovee p_{2m+1}) \arrow[u, "\mathsf{rbind}(\gamma_{2m}, \gamma_{2m})"]
\end{tikzcd}$$

Hence, ($\textsc{m}_2$) entails

$$\begin{aligned}
\gamma_{2m} &= \mathsf{rbind}(\gamma_{2m}, \gamma_{2m}) \\
&= \mathsf{rbind}(\Phi_{p_{2m-1}, p_{2m}}(\gamma_{2m-1}), \Phi_{q_{2m-1}, q_{2m}}(\gamma_{2m-1})) \\
&= \Phi_{q_{2m-1} \ovee p_{2m-1}, p_{2m} \owedge q_{2m}}(\mathsf{lbind}(\gamma_{2m-1}, \gamma_{2m-1})) \\
&= \Phi_{(p \owedge q)_{2m-1}, (p \owedge q)_{2m}}(\gamma_{2m-1})
\end{aligned}$$

which covers the case $n = 2m$; the case $n = 2m + 1$ follows by a symmetric argument involving ($\textsc{m}_2^{-1}$) or, alternatively, by invoking ($\textsc{inv}$).

The second statement is proven as follows.



∎

Hence, every cyclic star-autonomous category is equivalent to one which has *a strict negation* in the sense described in section 1.

A. Miscellaneous proofs

A.1. Lemma. *Let $(M, \mu, \mu_\circ)$ and $(N, \nu, \nu_\circ)$ be strong monoidal functors $(\mathcal{J}, \otimes, i) \longrightarrow (\mathcal{K}, \otimes, k)$, and ω be a natural isomorphism $M \longrightarrow N$ satsifying*

$$\begin{tikzcd}
M(p) \otimes M(q) \arrow[r, "\mu"] \arrow[d, "\omega_p \otimes \omega_q"'] & M(p \otimes q) \arrow[d, "\omega_{p \otimes q}"] \\
N(p) \otimes N(q) \arrow[r, "\nu"] & N(p \otimes q)
\end{tikzcd}$$

—then $\begin{tikzcd} k \arrow[r, "\mu_\circ"] \arrow[rr, bend right, "\nu_\circ"'] & M(i) \arrow[r, "\omega_i"] & N(i) \end{tikzcd}$ *also holds.*

Proof. It suffices to show that $\mu_\circ; \omega_j; \nu_\circ^{-1}$ is idempotent with respect to composition (since it is already known to be invertible). But, by the Eckmann-Hilton argument, it is equivalent to show that it is idempotent with respect to tensor (which is to say, in the arrow category,

$$(\mu_\circ; \omega_j; \nu_\circ^{-1}) \otimes (\mu_\circ; \omega_j; \nu_\circ^{-1}) \cong (\mu_\circ; \omega_j; \nu_\circ^{-1})$$

via the canonical isomorphism $k \otimes k \xrightarrow{\lambda=\rho} k$). This is demonstrated in the diagram below:

$$
\begin{tikzcd}
k \otimes k \arrow[r, "\mathrm{id} \otimes \mu_\circ"] \arrow[d, "\lambda = \rho"'] \arrow[rr, bend left, "\mu_\circ \otimes \mu_\circ"] \arrow[dr, phantom, "(\mathrm{N})"] & k \otimes M(i) \arrow[r, "\mu_\circ \otimes \mathrm{id}"] \arrow[d, "\lambda"] & M(i) \otimes M(i) \arrow[d, "\mu"] \arrow[ddd, bend left, "\omega_j \otimes \omega_j"] \\
k \arrow[r, "\mu_\circ"'] & M(i) \arrow[d, "\omega_j"'] \arrow[dr, phantom, "(\mathrm{N})"] & M(i \otimes i) \arrow[l, "M(\rho = \lambda)"'] \arrow[d, "\omega_{j\otimes j}"] \\
k \arrow[d, "\lambda^{-1} = \rho^{-1}"'] \arrow[dr, phantom, "(\mathrm{N})"] & N(i) \arrow[l, "\nu_\circ^{-1}"'] \arrow[d, "\rho^{-1}"] & N(i \otimes i) \arrow[l, "N(\rho = \lambda)"'] \arrow[d, "\nu^{-1}"] \\
k \otimes k & N(i) \otimes k \arrow[l, "\nu_\circ^{-1} \otimes \mathrm{id}"] & N(i) \otimes N(i) \arrow[l, "\mathrm{id} \otimes \nu_\circ^{-1}"] \arrow[ll, bend left, "\nu_\circ^{-1} \otimes \nu_\circ^{-1}"]
\end{tikzcd}
$$

the cells labelled (N) are naturality squares; the rightmost cell is a special case of the hypothesis; the topmost and bottommost cells are trivial; the remaining two squares are special cases of the monoidality of M and N; the outermost cell is the thing being proven.

■

A.2. LEMMA. *In any linearly distributive category,*

$$\alpha^{-1}_{q,s \ovee t,p} \; ; \; (\vec{\kappa}_{q,s,t} \; ; \; \psi \ovee \mathrm{id}_t \; ; \; \lambda_t) \owedge \mathrm{id}_p \; ; \; \omega' = \mathrm{id}_q \owedge (\overleftarrow{\kappa}_{s,t,p} \; ; \; \mathrm{id}_s \ovee \omega' \; ; \; \rho_s) \; ; \; \psi$$

holds for all $q \owedge s \xrightarrow{\psi} d$ *and* $t \owedge p \xrightarrow{\omega'} d$.

PROOF. In the diagram below: the pentagon and the two outermost triangles are among the axioms of a linearly distributive category; the central triangles are tautologies; the

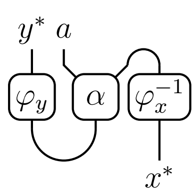
squares are all naturality squares.

$$
\begin{tikzcd}
 & (q \owedge (s \ovee t)) \owedge p \arrow[rrrr, "\alpha_{q,(s \ovee t),p}"] \arrow[d, "\vec{\kappa}_{q,s,t} \owedge \mathrm{id}_p"'] & & & & q \owedge ((s \ovee t) \owedge p) \arrow[d, "\mathrm{id}_q \owedge \overleftarrow{\kappa}_{s,t,p}"] & & \\
 & ((q \owedge s) \ovee t) \owedge p \arrow[dl, "(\psi \ovee \mathrm{id}_t) \owedge \mathrm{id}_p"'] \arrow[drr, "\overleftarrow{\kappa}_{(q \owedge s),t,p}"] & & & & q \owedge (s \ovee (t \owedge p)) \arrow[dll, "\vec{\kappa}_{q,s,(t \owedge p)}"'] \arrow[dr, "\mathrm{id}_q \owedge (\mathrm{id}_s \ovee \omega')"] & & \\
(d \ovee t) \owedge p \arrow[dr, "\overleftarrow{\kappa}_{d,t,p}"] \arrow[dd, "\lambda_t \owedge \mathrm{id}_p"'] & & & (q \owedge s) \ovee (t \owedge p) \arrow[dll, "\psi \ovee \mathrm{id}_{t \owedge p}"'] \arrow[dd, "\psi \ovee \omega'"] \arrow[drr, "\mathrm{id}_{q \owedge s} \ovee \omega'"] & & & q \owedge (s \ovee d) \arrow[dl, "\vec{\kappa}_{q,s,d}"'] \arrow[dd, "\mathrm{id}_q \owedge \rho_s"] & \\
 & d \ovee (t \owedge p) \arrow[dl, "\lambda_{t \owedge p}"] \arrow[drr, "\mathrm{id}_d \ovee \omega'"] & & & & (q \owedge s) \ovee d \arrow[dll, "\psi \ovee \mathrm{id}_d"'] \arrow[dr, "\rho_{q \owedge s}"] & & \\
t \owedge p \arrow[drrr, "\omega'"'] & & & d \ovee d \arrow[d, "\lambda_d = \rho_d"] & & & q \owedge s \arrow[dlll, "\psi"] & \\
 & & & d & & & &
\end{tikzcd}
$$

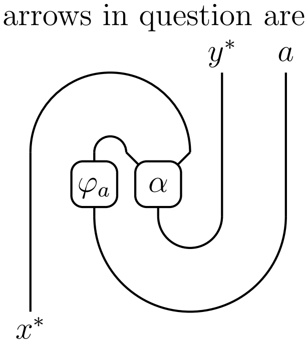
■

A.3. Lemma. *The two arrows* $(\langle r, q\rangle_f)^* \owedge \langle r, s\rangle_{\mathbb{a}} \longrightarrow (\langle s, q\rangle_f)^*$ *described in the proof of Theorem 3.1 are equal.*

Proof. We argue (semi)graphically: the two arrows in question are

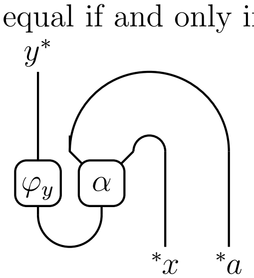


and

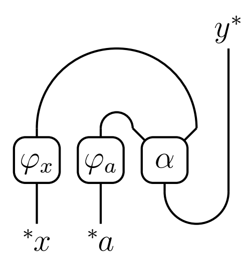


where $a = \langle r, s\rangle_{\mathbb{a}}$, $x = \langle s, q\rangle_f$, $y = \langle r, q\rangle_f$, α denotes the original action $a \owedge x \longrightarrow y$.

Clearly, these are equal if and only if



=



—but here the left-hand side represents

$$y^* \xrightarrow{\quad\varphi_y\quad} {}^*y \xrightarrow{\quad{}^*\alpha\quad} {}^*(a \owedge x) \xrightarrow{\quad\vartheta\quad} {}^*x \ovee {}^*a$$

which, by the naturality of φ, equals

$$y^* \xrightarrow{\quad{}^*\alpha\quad} (a \owedge x)^* \xrightarrow{\quad\varphi_{a\owedge x}\quad} {}^*(a \owedge x) \xrightarrow{\quad\vartheta\quad} {}^*x \ovee {}^*a$$

which in turn, by (T_2), equals

$$y^* \xrightarrow{\quad{}^*\alpha\quad} (a \owedge x)^* \xrightarrow{\quad\vartheta\quad} x^* \ovee a^* \xrightarrow{\quad\varphi_x \ovee \varphi_a\quad} {}^*x \ovee {}^*a$$

which is what the right-hand side represents. ∎

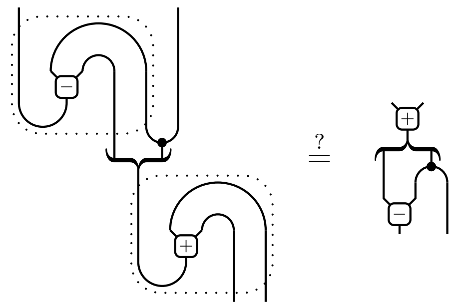
A.4. Lemma. *In any braided star-autonomous category, the following diagrams commute.*

$$\begin{tikzcd}
 & & (p \ovee q) \owedge r \arrow[dll, "\check{\beta}^{-1}_{p,q} \owedge \mathrm{id}_r"'] \arrow[drr, "\hat{\beta}_{p\ovee q,r}"] & & \\
(q \ovee p) \owedge r \arrow[ddd, "\overleftarrow{\kappa}_{q,p,r}"'] & & & & r \owedge (p \ovee q) \arrow[ddd, "\overrightarrow{\kappa}_{r,p,q}"] \\
 & & r \owedge (q \ovee p) \arrow[ull, dotted, "\hat{\beta}^{-1}_{q\ovee p,r}"] \arrow[urr, dotted, "\mathrm{id}_r \owedge \check{\beta}_{p,q}"'] & & \\
 & & q \ovee (r \owedge p) & & \\
q \ovee (p \owedge r) \arrow[urr, dotted, "\mathrm{id}_q \ovee \hat{\beta}_{r,p}"] \arrow[drr, "\check{\beta}_{q,p\owedge r}"'] & & & & (r \owedge p) \ovee q \arrow[ull, dotted, "\check{\beta}^{-1}_{q,r\owedge p}"'] \arrow[dll, "\hat{\beta}^{-1}_{r,p} \ovee \mathrm{id}_q"] \\
 & & (p \owedge r) \ovee q & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & r \owedge (q \ovee p) \arrow[dll, "\hat{\beta}_{q\ovee p,r}"'] \arrow[drr, "\mathrm{id}_r \owedge \check{\beta}^{-1}_{p,q}"] & & \\
(q \ovee p) \owedge r \arrow[ddd, "\overleftarrow{\kappa}_{q,p,r}"'] & & & & r \owedge (p \ovee q) \arrow[ddd, "\overrightarrow{\kappa}_{r,p,q}"] \\
 & & (p \ovee q) \owedge r \arrow[ull, dotted, "\check{\beta}_{p,q} \owedge \mathrm{id}_r"] \arrow[urr, dotted, "\hat{\beta}^{-1}_{p\ovee q,r}"'] & & \\
 & & (p \owedge r) \ovee q & & \\
q \ovee (p \owedge r) \arrow[urr, dotted, "\check{\beta}^{-1}_{q,p\owedge r}"] \arrow[drr, "\mathrm{id}_q \ovee \hat{\beta}^{-1}_{r,p}"'] & & & & (r \owedge p) \ovee q \arrow[ull, dotted, "\hat{\beta}_{r,p} \ovee \mathrm{id}_q"'] \arrow[dll, "\check{\beta}_{q,r\owedge p}"] \\
 & & q \ovee (r \owedge p) & &
\end{tikzcd}$$

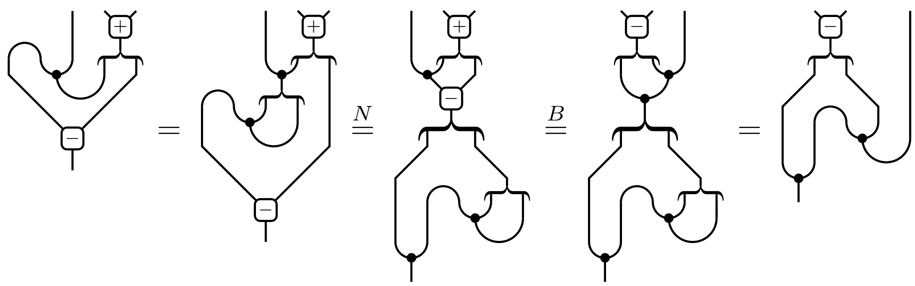
Proof. We prove only the first solid diagram; the second solid diagram follows by a symmetric argument, and the two dotted diagrams are obtained by attaching naturality squares.

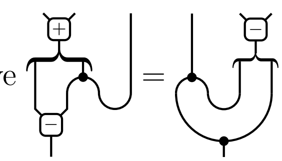
We work in the graphical calculus for *planar* star-autonomous categories; a box labelled with a plus sign denotes a component of $\hat{\beta}$; a box labelled with a minus sign denotes a component of $\hat{\beta}^{-1}$; the components of $\check{\beta}$ and $\check{\beta}^{-1}$ are defined by

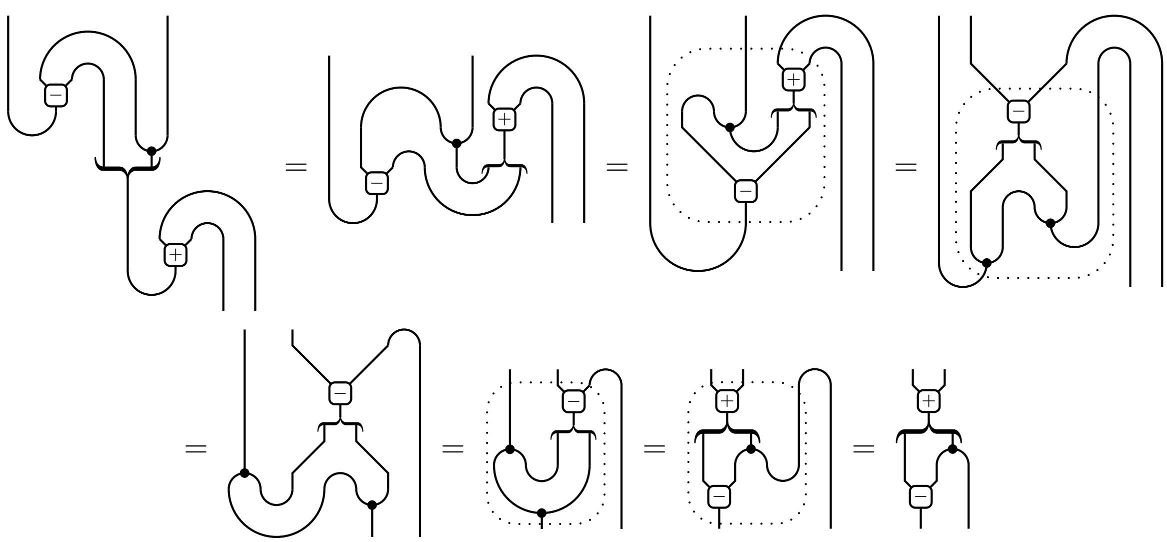
respectively; *switching links* are denoted by moustaches, and *non-switching links* by bullets; what we need to show is summarised below.



By naturality and (a variant of) the braiding axiom, we obtain



and, by an almost identical argument, one can also derive . Hence,



as desired. ∎

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